Physics 332 – Problem Set # 5

Solutions

1.) a.) Write the terms in the $\mathcal{L}$ that are quadratic in A_μ:

$$-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \quad + \quad \phi^*(eA^\mu)(eA_\mu)\phi$$

$$= \tfrac{1}{2}A_\nu\left(\partial^2 g^{\nu\lambda} - \partial^\nu\partial^\lambda + g^{\nu\lambda}e^2 2|\phi|^2\right)A_\lambda$$

so if we ignore $\partial^\lambda A_\lambda$

$$= \quad A_\nu\,(\partial^2 + m_A^2)\,g^{\nu\lambda}A_\lambda \qquad \text{where} \quad m_A^2 = 2e^2\langle|\phi|^2\rangle$$

b.) Now compute the contribution of A_μ to the effective potential for ϕ. Write

$$\phi \;=\; \varphi_{cl.} + \tfrac{1}{\sqrt{2}}(\sigma + i\pi)$$

assume real

$$m_A^2 = e^2 v^2$$

$$= \; v/\sqrt{2} + \tfrac{1}{\sqrt{2}}(\sigma + i\pi)$$

then

$$\mathcal{L} \;=\; A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu + \left(\tfrac{1}{\sqrt{2}}\right)^2\left|\partial_\mu(\sigma+i\pi) + ieA_\mu(v+\sigma+i\pi)\right|^2$$

$$- \;\frac{m^2}{2}|v+\sigma+i\pi|^2 - \;\frac{\lambda}{4!}\left(|v+\sigma+i\pi|^2\right)^2$$

$$= \quad A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu \quad + \tfrac{1}{2}(\partial_\mu\sigma)^2$$

$$+ \ \tfrac{1}{2}(\partial\pi \ + ev\,A_\mu)^2$$

$$- \left(\frac{m^2}{2}v^2 + \frac{\lambda}{24}v^4\right) \quad - (\text{lin } \text{in } \sigma)$$

$$- \quad \frac{m^2}{2}(\sigma^2+\pi^2) \ - \ \frac{\lambda}{4!}\cdot(6v^2\sigma^2 + 2v^2\pi^2)$$

$$- \ (\text{cubic, quartic in fields})$$

We would like to quantize so as to cancel the term $\partial_\mu\pi \cdot evA_\mu$

To do this, introduce the R_ξ gauge [see P+S section 21.1]

$$\int \mathcal{D}A \ = \ \int \mathcal{D}A \int \mathcal{D}\alpha \int d\omega \ \ e^{-i\int d^4x\ \omega^2/2} \ \delta(\omega - G\alpha) \ \det\frac{\delta G_\alpha}{\delta\alpha}$$

where $\quad G\alpha = \frac{1}{\sqrt{\xi}}(\partial_\mu A^\mu - \xi e v\pi)$

$$\frac{\delta G}{\delta\alpha} = \frac{1}{\sqrt{\xi}}\left(\tfrac{1}{e}\partial^2 + \xi e v^2\right) = \frac{1}{\sqrt{\xi}e}\left(\partial^2 + \xi e^2 v^2\right)$$

$$\int \mathcal{D}A = \int \mathcal{D}\alpha \ \cdot \ e^{i\int d^4x \left\{ -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \ + \ \partial_\mu A^\mu\cdot ev\pi \ - \frac{1}{2}\xi(ev)^2\pi^2 \right\}}$$

$$\cdot \ \det\left(\partial^2 + \xi(ev)^2\right)$$

Notice that the overall constant terms in the measure <u>do not</u> depend on $\underline{\underline{v}}$!

then

$$\int \mathcal{D}A \int d\phi \quad e^{i\int d^4x\, \mathcal{L}}$$

$$= \int \mathcal{D}A\, \mathcal{D}\sigma\, \mathcal{D}\pi \quad \exp\Big[i \int d^4x \Big\{ \tfrac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) A_\nu$$

$$+ \tfrac{1}{2} (ev)^2 A_\mu g^{\mu\nu} A_\nu \; + \; \tfrac{1}{2}\tfrac{1}{\xi} A_\mu \partial^\mu\partial^\nu A_\nu$$

$$+ \tfrac{1}{2}(\partial_\mu \pi)^2 \; - \; \tfrac{1}{2}\xi (ev)^2 \pi^2 - \tfrac{1}{2}\left(\tfrac{\lambda v^2}{6} + m^2\right)\pi^2$$

$$+ \; \tfrac{1}{2}(\partial_\mu \sigma)^2 \; - \; \tfrac{1}{2}\left(\tfrac{3\lambda}{6} v^2 + m^2\right)\sigma^2 \; - \left(\tfrac{m^2}{2} v^2 + \tfrac{\lambda}{24} v^4\right)$$

$$+ (\text{linear in } \sigma) \; + \; \text{cubic, quartic} \Big\} \Big]$$

$$\cdot \det\left(\partial^2 + \xi (ev)^2\right)$$

c) We might as well do the integral. Note that the transverse and longitudinal components of A_μ separate:

$$\tfrac{1}{2}(ev)^2 A_\mu g^{\mu\nu} A_\nu = \tfrac{1}{2} m_A^2 \left[A_\mu \left(g^{\mu\nu} - \tfrac{\partial^\mu\partial^\nu}{\partial^2}\right) A_\nu \; + \; A_\mu \tfrac{\partial^\mu\partial^\nu}{\partial^2} A_\nu \right]$$

then the quadratic terms in A are:

$$\tfrac{1}{2} A_\mu \left[\left(g^{\mu\nu} - \tfrac{\partial^\mu\partial^\nu}{\partial^2}\right)(\partial^2 + m_A^2) \; + \; \tfrac{\partial^\mu\partial^\nu}{\partial^2} \cdot \tfrac{1}{\xi}(\partial^2 + \xi m_A^2) \right] A_\nu$$

so, the value of the integral is:

$\int \mathcal{D}A \int d\phi \; e^{i\int \mathcal{L}}$

$$= e^{-i\int d^4x \, (m^2/2 \, v^2 + \lambda/24 \, v^4) \; + \; (\text{lin.\ in } \sigma)}$$

$$\cdot \left[\det(\partial^2 + m_A^2)\right]^{-3/2} \left[\det(\partial^2 + \xi m_A^2)\right]^{-1/2} \left[\det(\partial^2 + \xi m_A^2)\right]^{+1}$$

from A, log

$$\cdot \left(\det\left[\partial^2 + \xi m_A^2 + \left(\frac{\lambda v^2}{6} + m^2\right)\right]\right)^{-1/2} \qquad \text{from } \pi$$

$$\cdot \det\left[\partial^2 + \left(\frac{3\lambda v^2}{6} + m^2\right)\right]^{-1/2} \qquad \text{from } \sigma$$

From here, we see ~~that~~

$$V_{eff}(v) = \frac{m^2}{2} v^2 + \frac{\lambda}{24} v^4$$

$$+ i \frac{1}{\text{Vol.}} \cdot \Big[-\frac{3}{2} \log\det(\partial^2 + m_A^2) + \frac{1}{2} \log\det(\partial^2 + \xi m_A^2)$$

$$- \frac{1}{2} \log\det\left(\partial^2 + \xi m_A^2 + \left(\frac{\lambda v^2}{6} + m^2\right)\right)$$

$$- \frac{1}{2} \log\det\left(\partial^2 + \left(\frac{3\lambda v^2}{6} + m^2\right)\right) \Big]$$

In perturbation theory, at the zeroth order minimum of V_{eff} for $m^2 < 0$

$$-(-m^2)v + \frac{\lambda}{6} v^3 = 0 \quad \text{or} \quad \left(\frac{\lambda v^2}{6} + m^2\right) = 0$$

then the ξ dependence of this expression cancels.

In fact, the energy at the minimum and the mass of the σ and A are ξ-invariant to all orders.

For simplicity, I will set $\xi = 0$ from here on. The functional determinants have the values

$$\frac{1}{\text{Vol}} \log \det(\partial^2 + m^2) = -i \cdot \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} (m^2)^{d/2}$$

$$d = 4-\varepsilon \qquad = -i \frac{(m^2)^2}{2 \cdot (4\pi)^2} \cdot \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log m^2 + \tfrac{3}{2} \right)$$

the divergent terms are:

$$V_{eff} = + \frac{1}{2(4\pi)^2} \cdot \frac{2}{\epsilon} \left\{ -\frac{3}{2} (ev)^4 - \frac{1}{2} \left(\frac{\lambda v^2}{6} + m^2 \right)^2 - \frac{1}{2} \left(\frac{\lambda v^2}{2} + m^2 \right)^2 \right\}$$

$$= \frac{1}{2(4\pi)^2} \frac{2}{\epsilon} \cdot \left\{ (\text{const}) + v^2 \left[-\frac{\lambda}{6} - \frac{\lambda}{2} \right] m^2 + v^4 \left[-\frac{3}{2} e^4 - \frac{\lambda^2}{72} - \frac{\lambda^2}{8} \right] \right\}$$

$$= \frac{1}{2(4\pi)^2} \frac{2}{\epsilon} \left\{ (\text{const}) + v^2 \left[-\frac{2}{3} \lambda m^2 \right] + v^4 \left[-\frac{3}{2} e^4 - \frac{5}{36} \lambda^2 \right] \right\}$$

these divergences are to be cancelled by the counterterms.

$$V_{eff} = \quad \cdots \quad + \frac{1}{2}\delta m\, v^2 + \frac{1}{24}\delta\lambda\, v^4$$

$$\delta m = \frac{\lambda m^2}{3(4\pi)^2}\cdot\frac{2}{\epsilon} \qquad \delta\lambda = \frac{1}{(4\pi)^2}\left[\,18\, e^4 + \frac{5}{3}\lambda^2\right]\cdot\frac{2}{\epsilon}$$

Making this subtraction (eliminating at the same time $(-\gamma + \log 4\pi)$'s, we find

$$V_{eff} = \frac{m^2}{2}v^2 + \frac{\lambda}{24}v^4$$

$$+ \frac{1}{4(4\pi)^2}\cdot\Big\{\; 3(ev)^4\left(\log\frac{(ev)^2}{M^2} - \frac{3}{2}\right)$$

$$+ \left(\frac{\lambda v}{6} + m^2\right)^2\left(\log\frac{\lambda v/6 + m^2}{M^2} - \frac{3}{2}\right)$$

$$+ \left(\frac{\lambda v}{2} + m^2\right)^2\left(\log\frac{\lambda v/2 + m^2}{M^2} - \frac{3}{2}\right)\Big\}$$

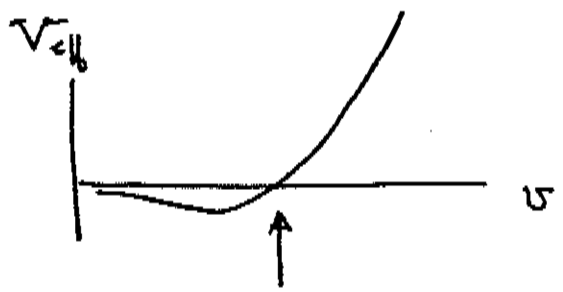
d) let $m^2 \to 0$

$$V_{eff} = \frac{\lambda v^4}{24} - \frac{3}{4(4\pi)^2} e^4 v^4 \left(\log \frac{M^2}{(ev)^2} + \frac{3}{2} \right)$$

$+$ small if $\lambda \sim e^4$

for $\log M^2/(ev)^2 > -3/2 \qquad ev < M e^{3/4}$

the second term is negative. For sufficiently small v, the coefficient of v^4 becomes negative, so the potential takes the form



$$\frac{\lambda}{24} - \frac{3}{4(4\pi)^2} e^4 \left(\log \frac{M^2}{(ev)^2} + \frac{3}{2} \right) = 0$$

$$\log M^2/(ev)^2 = -\frac{3}{2} + \frac{(4\pi)^2}{18} \frac{\lambda}{e^4}$$

$$ev = M \exp\left[-\frac{4\pi^2}{9} \frac{\lambda}{e^4} + \frac{3}{4} \right]$$

e.) Now compute the renormalization group β-functions for this theory. To do this, we need

δ_Z scalar field strength counterterm

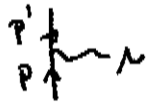
δ_3 A_μ field strength counterterm

δ_λ

δ_Z and δ_λ are gauge-dependent. I will compute them for general ξ, but it is OK to compute them just for $\xi = 0$ or $\xi = 1$.

The β functions are gauge-independent.

I will use complex fields $\phi, \phi^\dagger$. The Feynman rules are

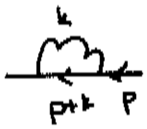
 $= -ie\,(p+p')^\mu$  $= 2ie^2 g^{\mu\nu}$

 $= -i\,\frac{2}{3}\,\lambda$

Begin with

$ip^2\delta_Z$

 and  are mass renormalizations only with no contribution to δ_Z



$$= (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \, (2p+k)^\mu \frac{i}{(p+k)^2} (2p+k)^\nu \quad \frac{-i}{k^2}$$

$$\cdot \left[g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-\xi) \right]$$

$$= (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2 (k^2)^2} \left((2p+k)^2 k^2 - [k\cdot(2p+k)]^2 (1-\xi) \right)$$

$$= -e^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2 (k^2)^2} \Big[k^4 + 4k\cdot p\, k^2 + 4p^2 k^2$$

$$- (1-\xi)\left(k^4 + 4k\cdot p\, k^2 + 4(k\cdot p)^2\right) \Big]$$

combine denominators

$$\mathbb{K} = k + xp \qquad k = \mathbb{K} - xp$$

$$k^2 = \mathbb{K}^2 - 2xp\cdot\mathbb{K} + x^2 p^2$$

$$k\cdot p = \mathbb{K}\cdot p - xp^2$$

$$= -e^2 \int \frac{d^d \mathbb{K}}{(2\pi)^d} \int dx \; \frac{2(1-x)}{[\mathbb{K}^2 + x(1-x)p^2]^3}$$

$$\cdot \left\{ \xi \left[k^2 (k^2 + 4k\cdot p) \right] + 4\left[p^2 k^2 - (1-\xi)(k\cdot p)^2 \right] \right\}$$

$$= -e^2 \int \frac{d^d \mathbb{K}}{(2\pi)^d} \frac{2(1-x)}{[\mathbb{K}^2 + x(1-x)p^2]^3}$$

$$\Big\{ \xi \Big[(\mathbb{K}^2)^2 - 4x\, p\cdot\mathbb{K}\, \mathbb{K}^2 + 4x^2 (p\cdot\mathbb{K})^2 + 2x^2 \mathbb{K}^2 p^2 + \cdots$$

$$+ 4\mathbb{K}^2\, \mathbb{K}\cdot p - \cancel{8}x\, p\cdot\mathbb{K}\, \mathbb{K}\cdot p - 4x\, \mathbb{K}^2 p^2 + \cdots \Big]$$

$$+ 4\left[p^2 \mathbb{K}^2 - (1-\xi)(\mathbb{K}\cdot p)^2 \right] \Big\}$$

$$= -e^2 \int_0^1 dx \quad 2(1-x) \quad \frac{i}{(4\pi)^{d/2}} \quad \frac{1}{[-x(1-x)p^2]^{2-d/2}} \quad \frac{1}{\Gamma(3)}$$

$$\Big\{ \quad \xi \Big[-\frac{d(d+2)}{4} \quad \Gamma(1-d/2) \quad (-x(1-x)p^2)$$

$$+ \quad \Gamma(2-d/2) \quad \Big[4x^2 \cdot \tfrac{1}{2} \cdot p^2 + 2x^2 \tfrac{d}{2} p^2$$

$$- 8x \; \tfrac{1}{2} p^2 - 4x \; \tfrac{d}{2} p^2 \Big]$$

$$+ 4\,\Gamma(2-d/2) \Big[\; d/2 \; p^2 - (1-\xi) \tfrac{1}{2} p^2 \Big] \quad \Big\}$$

near $\quad d = 4-\varepsilon \qquad \Gamma(2-d/2) \cong \frac{2}{\epsilon} \qquad \Gamma(1-d/2) \cong \frac{\Gamma(2-d/2)}{1-d/2} \cong -\frac{2}{\epsilon}$

$$= \; - \frac{ie^2}{(4\pi)^{d/2}} \frac{2}{\epsilon} \int_0^1 dx \quad 2(1-x) \cdot \frac{1}{2} \cdot \quad p^2$$

$$\cdot \Big\{ \quad \xi \quad \Big[-6x(1-x) + 2x^2 + 4x^2 - 4x - 8x \Big]$$

$$+ \quad 4 \quad \Big[\; 2 - \tfrac{1}{2} + \xi/2 \; \Big] \; \Big\}$$

$$= \; - \frac{ie^2}{(4\pi)^{d/2}} \; \frac{2}{\epsilon} \; p^2 \quad \int_0^1 dx \quad (1-x)$$

$$\cdot \Big\{ \quad 6 \; + \; \xi \; \Big[\; 2 - 18x + 12x^2 \Big] \; \Big\}$$

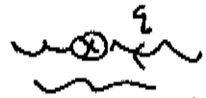
$$= \; \frac{-ie^2}{(4\pi)^2} \; \frac{2}{\epsilon} \; p^2 \; \Big(3 + \xi \Big[\underbrace{1 - \frac{18}{6} + \frac{12}{12}}_{-1} \Big] \Big)$$

So

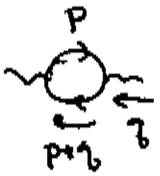
$$= \frac{-ie^2}{(4\pi)^2}\, p^2 \cdot (3-\xi) \cdot \frac{2}{\epsilon} \qquad\qquad \frac{2}{\epsilon} \sim \log \Lambda^2$$

$$+ \; ip^2 \delta_Z$$

so $$\delta_Z = \frac{e^2}{(4\pi)^2}(3-\xi) \log \Lambda^2/M^2$$

$=$  $+$  $+$ 

$$-i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\delta_3$$



$$= (-ie)^2 \int \frac{d^dp}{(2\pi)^d}\, (2p+q)^\mu \frac{i}{(p+q)^2}\, (2p+q)^\nu\, \frac{i}{p^2}$$

$$= e^2 \int_0^1 dx \int \frac{d^d\mathbb{P}}{(2\pi)^d}\, \frac{(2\mathbb{P} + (1-2x)q)^\mu (2\mathbb{P} + (1-2x)q)^\nu}{[\mathbb{P}^2 + x(1-x)q^2]^2}$$

using $\mathbb{P} = p + xq$ $\qquad 2p+q = 2\mathbb{P} + (1-2x)q$

$\qquad\qquad\qquad\qquad\quad p+q = \mathbb{P} + (1-x)q$

$$= e^2 \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

$$\Big\{ -4 \cdot \tfrac{1}{2} g^{\mu\nu}\, \Gamma(1-\tfrac{d}{2})\, (-x(1-x)q^2)$$

$$+ \;\Gamma(2-\tfrac{d}{2})\, q^\mu q^\nu\, (1-2x)^2 \Big\}$$

 $= \quad 2ie^2 g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d} \frac{+i}{p^2}$

$$= -2ie^2 g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 (p+q)^2} (p+q)^2$$

$$= -2ie^2 g^{\mu\nu} \int \frac{d^dP}{(2\pi)^d} \int dx \frac{[P + (1-x)q]^2}{[P^2 + x(1-x)q^2]^2}$$

or symmetrize the numerator in $x \leftrightarrow (1-x)$

$$= -2e^2 g^{\mu\nu} \int dx \frac{i}{(4\pi)^{d/2}} \Big\{ -\Gamma(1-\tfrac{d}{2})\, \tfrac{d}{2}\, (-x(1-x)q^2)$$

$$+ \tfrac{1}{2}[(1-x)^2 + x^2]\, q^2\, \Gamma(2-\tfrac{d}{2}) \Big\} \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

in all

 + 

$$= \frac{ie^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{(-x(1-x)q^2)^{2-d/2}}$$

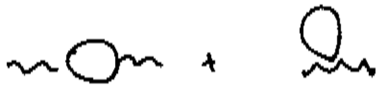
$$\cdot \Big\{ \Gamma(1-\tfrac{d}{2})\, (-x(1-x)q^2)\, (-2+d)$$

$$+ \Gamma(2-\tfrac{d}{2})\, (1-2x)^2\, q^\mu q^\nu$$

$$- \Gamma(2-\tfrac{d}{2})\, q^2 g^{\mu\nu}\, (1-2x+2x^2) \Big\}$$

$$\Gamma(1-\tfrac{d}{2})(-2+d) = -2\,\Gamma(2-\tfrac{d}{2})$$

so



$$= \frac{ie^2}{(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 dx \, \frac{1}{[-x(1-x)q^2]^{2-d/2}}$$

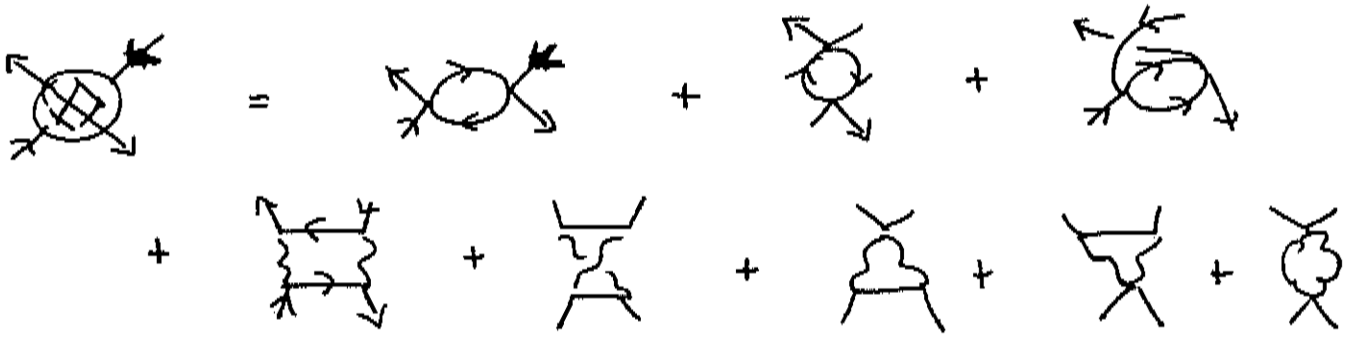
$$\left\{ -q^2 g^{\mu\nu} (1-4x+4x^2) + q^\mu q^\nu (1-2x)^2 \right\}$$

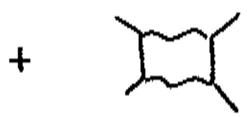
$$\cong -\frac{ie^2}{(4\pi)^2} \cdot \frac{2}{\epsilon} \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 dx \, \underbrace{(1-4x+4x^2)}_{1 - \frac{4}{2} + \frac{4}{3} \, = \, \frac{1}{3}}$$

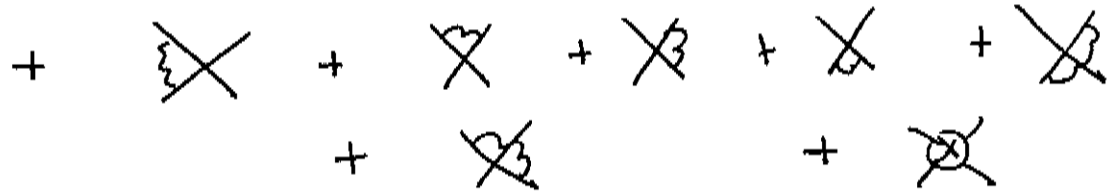
so

$$\delta_3 = -\frac{1}{3} \frac{e^2}{(4\pi)^2} \log \Lambda^2/M^2$$

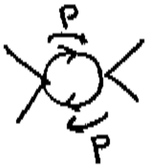
Finally, we need to compute δ_λ. There are 3 sets of loop diagrams:



 + (similar set in t-channel)



We can compute all of these diagrams with all external momenta set $= 0$



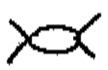
$$= \left(-i\tfrac{2}{3}\lambda\right)^2 \int \frac{d^dp}{(2\pi)^d}\,\frac{i}{p^2}\,\frac{i}{p^2}$$

$$= \frac{4}{9}\lambda^2\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2) = i\,\frac{4}{9}\lambda^2\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}$$

$$= \left(-i\tfrac{2}{3}\lambda\right)^2 \int \frac{d^dp}{(2\pi)^d}\left(\frac{i}{p^2}\right)^2 = i\,\frac{4}{9}\lambda^2\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}$$

$$= \left(-i\tfrac{2}{3}\lambda\right)^2\cdot\frac{1}{2}\cdot\int \frac{d^dp}{(2\pi)^d}\left(\frac{i}{p^2}\right)^2 = i\,\frac{2}{9}\lambda^2\,\frac{1}{(4\pi)^2}\,\frac{2}{\epsilon}$$

in all

 +  + 

$$= i\,\frac{10}{9}\lambda^2\,\frac{1}{(4\pi)^2}\cdot\frac{2}{\epsilon}$$

$$= (-ie)^4 \int \frac{d^dp}{(2\pi)^d}\left(\frac{i}{p^2}\right)^2\left(\frac{-i}{p^2}\right)^2$$

$$\cdot\, p^\mu p^\nu \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right)\cdot p^\lambda p^\sigma$$

$$= (-ie)^4 \int \frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^4}\cdot\,\xi^2\,(p^2)^2$$

$$= \frac{ie^4}{(4\pi)^2}\,\Gamma(2-d/2)\cdot\xi^2$$

$$= -ie^4 \int \frac{d^dp}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)^2$$

$$p^\mu p^\nu \left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right)(-p^\lambda)(-p^\sigma)$$

$$= \frac{ie^4}{(4\pi)^2}\,\Gamma(2-d/2)\;\xi^2$$

$$= 2ie^2 g^{\mu\nu} (-ie)^2 \int \frac{d^dp}{(2\pi)^d}\,\frac{i}{p^2}\left(\frac{-i}{p^2}\right)^2$$

$$\cdot\left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right) p^\lambda p^\sigma$$

$$= -2e^4\,\frac{1}{(4\pi)^2}\,\Gamma(2-d/2)\;\xi^2$$

=

$$= \frac{1}{2}\cdot 2ie^2 g^{\mu\nu}\;\; 2ie^2 g^{\lambda\sigma} \int \frac{d^dp}{(2\pi)^d}\left(\frac{-i}{p^2}\right)^2$$

$$\left(g^{\mu\lambda} - (1-\xi)\frac{p^\mu p^\lambda}{p^2}\right)\left(g^{\nu\sigma} - (1-\xi)\frac{p^\nu p^\sigma}{p^2}\right)$$

$$= 2e^4\cdot(3+\xi^2)\,\frac{i}{(4\pi)^2}\cdot\Gamma(2-d/2)$$

 +  +  +  + 

$$= e^4\,\frac{i}{(4\pi)^2}\,\frac{2}{\epsilon}\cdot\left(2\xi^2 - 4\xi^2 + 6 + 2\xi^2\right)$$

the t channel works similarly, so

 + ... +  + ... $= 12 \cdot \frac{ie^4}{(4\pi)^2} \cdot \frac{2}{\epsilon}$

$p \downarrow$  $= (-i\tfrac{2}{3}\lambda)(-ie)^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)$

$$\cdot\, p^\mu p^\lambda \left[g_{\mu\nu} - (1-\xi)\frac{p_\nu p_\lambda}{p^2} \right]$$

$$= -\tfrac{2}{3}e^2\lambda \cdot \xi \cdot \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2)$$

$=$ 

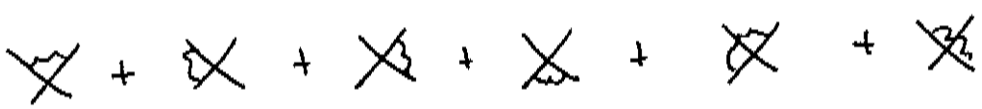
However

p  $= (-i\tfrac{2}{3}\lambda)(-ie)^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{i}{p^2}\right)^2 \left(\frac{-i}{p^2}\right)$

$$\cdot\, p^\mu \cdot (-p^\nu) \cdot \left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2} \right]$$

$$= +\tfrac{2}{3}e^2\lambda \cdot \xi \cdot \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2)$$

so



$$= -2\xi \cdot \tfrac{2}{3}e^2\lambda \, \frac{1}{(4\pi)^2} \cdot \frac{2}{\epsilon}$$

$\Rightarrow$ all: $\quad \cancel{\times} = -\frac{2}{3} i\,\delta_\lambda$

$$\frac{2}{3}\delta_\lambda = \left[\frac{10}{9}\frac{\lambda^2}{(4\pi)^2} - \frac{2}{3}\cdot 2\xi\frac{e^2\lambda}{(4\pi)^2} + 12\frac{e^4}{(4\pi)^2}\right]\lg \Lambda^2/M^2$$

then $$\delta_\lambda = \left[\frac{5}{3}\frac{\lambda^2}{4\pi^2} - 2\xi\frac{e^2\lambda}{(4\pi)^2} + 18\frac{e^4}{(4\pi)^2}\right]\lg \Lambda^2/M^2$$

$$\beta_e = M\frac{\partial}{\partial M}\left(-e\delta_1 + e\delta_2 + \tfrac{1}{2}e\delta_3\right)$$

$$= (\text{Abelian gge theory}) = M\frac{\partial}{\partial M}\left(\tfrac{1}{2}e\delta_3\right)$$

$$= \frac{1}{3}\frac{e^3}{(4\pi)^2} = \frac{e^3}{48\pi^2}$$

$$\beta_\lambda = M\frac{\partial}{\partial M}\left[-\delta_\lambda + 2\delta_Z\lambda\right]$$

$$= \frac{1}{(4\pi)^2}\left\{\frac{10}{3}\lambda^2 - 4\xi e^2\lambda + 36e^4 - 4(3-\xi)e^2\lambda\right\}$$

$$= \frac{1}{(4\pi)^2}\left\{\frac{10}{3}\lambda^2 - 12e^2\lambda + 36e^4\right\}$$

$$= \frac{1}{24\pi^2}\left\{5\lambda^2 - 18e^2\lambda + 54e^4\right\}$$

Note that this expression is independent of ξ. If we set $\xi = 0$ in δ_λ, we find:

$$\delta_\lambda\ (\xi = 0) = \left[\frac{5}{3}\lambda^2 + 18\, e^4\right] \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

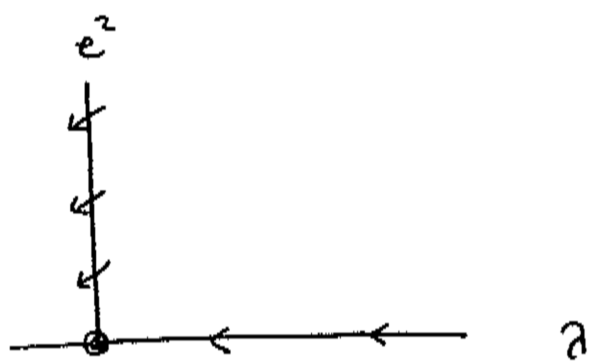
in agreement w. p. 6

Now look at flows of (λ, e^2) to the IR

e^2 always flows to 0 smoothly.

λ flows to 0 for $e^2 = 0$; however, on the line

$\lambda = 0$, λ flows to <u>negative</u> values:



$$\frac{d}{d\log q}\left(\frac{\lambda}{e^2}\right) = \frac{1}{e^2}\beta_\lambda - 2\frac{\lambda}{e^3}\beta_e$$

$$= \frac{1}{(4\pi)^2}\left\{ \frac{10}{3}\lambda^2/e^2 - 12\lambda + 36\, e^2 - \frac{2}{3}\lambda \right\}$$

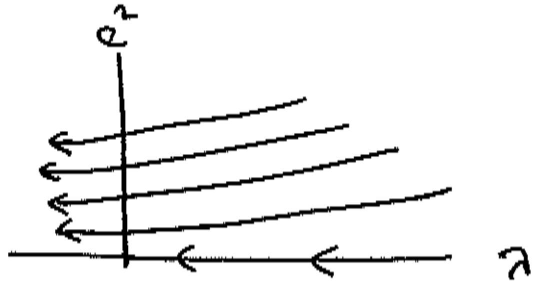
$$= \frac{e^2}{(4\pi)^2}\left\{ \frac{10}{3}\left(\lambda/e^2\right)^2 - \frac{38}{3}\left(\lambda/e^2\right) + 36 \right\}$$

$$= \frac{e^2}{(4\pi)^2}\left\{ \frac{10}{3}\left(\lambda/e^2 - \frac{19}{10}\right)^2 + \underbrace{36 - \frac{10}{3}\cdot\left(\frac{19}{10}\right)^2}_{>0} \right\}$$

$$> 0$$

so as $q \to 0$ λ/e^2 decreases, so we cannot set to $e^2 = 0$ $\lambda > 0$ from $e^2 > 0$

then, the flows are:



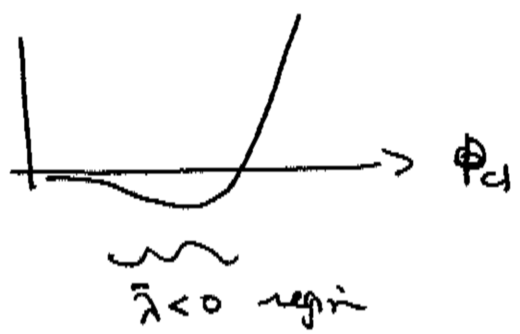
into the region $\lambda < 0$

f.) Then $\bar{\lambda}(\varphi_{cl})$ becomes negative for some φ_{cl}.

The resummed effective potential for $m^2 = 0$

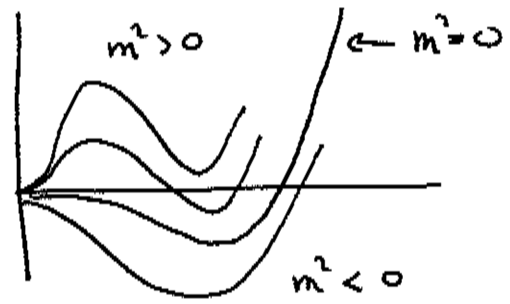
$$V_{eff} = \frac{1}{6} \bar{\lambda}(\varphi_{cl}) \cdot Z^2(\varphi_{cl}) \cdot \varphi_{cl}^4$$
$$+ \text{(small corrections)}$$

has the shape

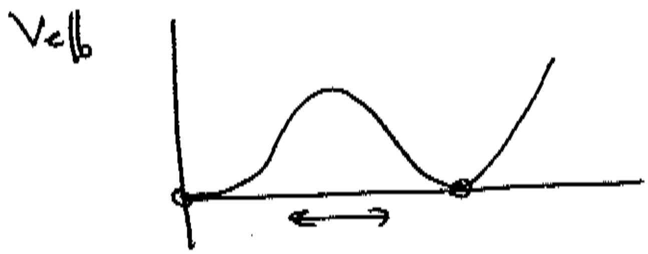


and has a symmetry-breaking minimum.

g.) Add back $m^2 \neq 0$. The successive effective potentials for $m^2 \neq 0$ have the form



at the m^2 st.



we have a 1^{st} order phase transition

For more details of this theory, see
S. Coleman and E. Weinberg PRD $\underline{7}$ 1888 (1973)

2.) a.) It is not so hard to extend this analysis to the GSW weak-interaction model. Using the gauge $\xi = 0$ and ignoring contributions from Higgs boson loops (which are proportional to λ^2), we find

$$V_{eff} = -\frac{\mu^2}{2} v^2 + \frac{\lambda}{4} v^4$$

$$+ i \frac{1}{(\mathrm{Vol})} \cdot \Big[-\frac{3}{2} \cdot 2 \cdot \log \det (\partial^2 + m_W^2)$$

$$- \frac{3}{2} \log \det (\partial^2 + m_Z^2) \Big]$$

where $m_W^2 = \dfrac{g^2 v^2}{4}$ $\qquad m_Z^2 = \dfrac{(g^2 + g'^2) v^2}{4}$

$=$ (after minimal subtraction)

$$-\mu^2 \frac{v^2}{2} + \frac{\lambda}{4} v^4$$

$$+ \frac{3}{4} \frac{1}{(4\pi)^2} \cdot \Big\{ 2 \left(\frac{g^2 v^2}{4}\right)^2 \left(\log \frac{g^2 v^2}{4M^2} - 3/2\right)$$

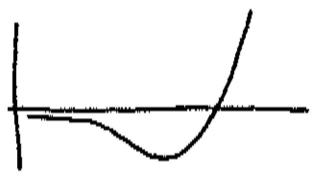
$$+ \left(\frac{(g^2 + g'^2) v^2}{4}\right)^2 \left(\log \frac{(g^2 + g'^2) v^2}{4M^2} - 3/2\right) \Big\}$$

b.) This effective potential is of the form

$$V_{eff} = -\mu^2 \frac{v^2}{2} + \frac{a}{4} \frac{g^4}{(4\pi)^2} v^4 \left(\log \frac{g^2 v^2}{M^2} + c \right)$$

and, to simplify the analysis, we could ~~even~~ absorb c into the definition of M. For $\mu^2 = 0$, we have then

$$V_{eff} = \frac{a}{4} \frac{g^4}{(4\pi)^2} v^4 \log \frac{g^2 v^2}{M^2}$$



and, turning on μ^2, we have a first-order phase transition, as on p. 20.

c.) The minimum of the potential is given by

$$\frac{\partial V_{eff}}{\partial v} = 0 \qquad \text{let } -\mu^2 = m^2$$

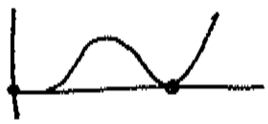
$$0 = m^2 v + a \frac{g^4}{(4\pi)^2} v^3 \log \frac{g^2 v^2}{M^2}$$

$$+ \frac{a}{2} \frac{g^4}{(4\pi)^2} v^4 \cdot \frac{1}{v}$$

or

$$0 = m^2 + a \frac{g^4}{(4\pi)^2} v^2 \left(\log \frac{g^2 v^2}{M^2} + \frac{1}{2} \right)$$

The condition for a 1st order phase transition is that the symmetry-breaking minimum should be at $V_{eff} = 0$



$$0 = m^2 \frac{v^2}{2} + \frac{a}{4} \frac{g^4}{(4\pi)^2} v^4 \log \frac{g^2 v^2}{M^2}$$

$$m^2 = -\frac{a}{2} \frac{g^4}{(4\pi)^2} v^2 \log \frac{g^2 v^2}{M^2} \quad > 0$$

Then the condition at the top of the page becomes.

$$0 = a \frac{g^4}{(4\pi)^2} v^2 \left(\frac{1}{2} \log \frac{g^2 v^2}{M^2} + \frac{1}{2} \right)$$

or $$\log \frac{g^2 v^2}{M^2} = -1$$

at this point:

$$\frac{\partial^2 V_{eff}}{\partial v^2} = m_h^2 = m^2 + 3a \frac{g^4}{(4\pi)^2} v^2 \left(\log \frac{g^2 v^2}{M^2} + \frac{1}{2} \right) + 2a \frac{g^4}{(4\pi)^2} v^2$$

$$m_h^2 = +\frac{a}{2}\frac{g^4 v^2}{(4\pi)^2} + 3a\frac{g^4}{(4\pi)^2}v^2\left(-1+\frac{1}{2}\right)$$
$$+ 2a\frac{g^4}{(4\pi)^2}v^2$$

$$= a\frac{g^4v^2}{(4\pi)^2}\cdot\left(\frac{1}{2}-\frac{3}{2}+2\right) = a\frac{g^4v^2}{(4\pi)^2}$$

from p. 21

$$\frac{a}{4}\frac{g^4}{(4\pi)^2} = \frac{1}{(4\pi)^2}\cdot\frac{1}{4}\cdot\left\{\frac{6}{16}g^4 + \frac{3}{16}(g^2+g'^2)^2\right\}$$

$$\left(\frac{g'}{g}\right)^2 = \tan^2\theta_w$$
$$\frac{g^2+g'^2}{g^2} = \frac{1}{\cos^2\theta_w}$$

$$\frac{ag^4}{(4\pi)^2} = \frac{1}{(4\pi)^2}\frac{3}{16}\left(2+\frac{1}{\cos^2\theta_w}\right)g^4$$

so

$$m_h^2 = \frac{g^2}{(4\pi)^2}\cdot\frac{3}{4}\left(2+\frac{1}{\cos^2\theta_w}\right)\left(\frac{g^2v^2}{4}\right)$$

$$= \frac{3}{16\pi}\left(2+\frac{1}{\cos^2\theta_w}\right)\alpha_w m_w^2$$

with $\alpha_w = \frac{g^2}{4\pi} = \frac{1}{30}$, $m_W = 80$ GeV, $\cos\theta_w = .77$

$$m_h^2 = \frac{3}{16\pi}\alpha_w \left(2 + \frac{1}{\cos^4\theta_w}\right) m_W^2$$

evaluates to

$$m_h = 6.8 \text{ GeV}$$

This is the Linde - Weinberg bound. For $m^2 = 0$, the minimum is at $\log \frac{g^2 v^2}{M^2} = -\frac{1}{2}$ and

$$m_h^2 = 2a \frac{g^4}{(4\pi)^2} v^2 = \frac{3}{8\pi}\alpha_w \left(2 + \frac{1}{\cos^4\theta_w}\right) m_W^2$$

or $m_h = 9.7$ GeV

the "Weinberg - Gildener value".

In fact, $m_h > 114$ GeV.

d.) Now add the contributions of the top quark and the Higgs bosons.

$$V_{eff} = (p.21)$$

$$+ i\left(\frac{1}{Vol}\right)\Big[-\frac{1}{2} \log\det\left(\partial^2 + (3\lambda v^2 - \mu^2)\right)$$

$$- \frac{3}{2} \log\det\left(\partial^2 + (\lambda v^2 - \mu^2)\right)$$

$$+ 3 \log\det\left(i\not\partial - m_t\right) \Big]$$

where $m_t = \frac{\lambda_t v}{\sqrt{2}}$, 3 is a color factor,

and the + sign is for fermions.

$$\log\det(i\not\partial - m_t) = 2 \log\det(\partial^2 + m_t^2)$$

evaluating these determinants in the same way as before and setting $\lambda v^2 = \mu^2$, we find a $\log v^2/M^2$ term:

$$V_{eff} = \cdots + \frac{1}{4}\frac{1}{(4\pi)^2}\Big[6 m_W^4 + 3 m_Z^4 + m_h^4$$

$$- 12 m_t^4 \Big]\left(\log \frac{g^2 v^2}{M^2} + \cdots\right)$$

This decreases the coefficient of the log and brings us back to the "normal" situation in which there is no radiative symmetry breaking. However, when m_t becomes large, the asymptotic behavior of the potential is

$$V_{eff} = - b \frac{v^4}{(4\pi)^2} \log v^2/M^2$$

and there is an instability to escape to $v^2 \to \infty$.

To analyze this situation, look at the leading pieces of each term of V_{eff} for $v \to \infty$

$$V_{eff} = \cdots + \frac{1}{4}\frac{1}{(4\pi)^2} \Big\{ 6\left(\frac{g^2 v^2}{4}\right)^2 + 3\left(\frac{(g^2+g'^2)v^2}{4}\right)^2$$

σ and π contribution →
$$+ (3\lambda v^2)^2 + 3(\lambda v^2)^2$$

$$- 12\left(\frac{\lambda_t^2 v^2}{2}\right)^2 \Big\} \log \frac{v^2}{M^2}$$

the term in brackets is $(m_h^2 = 2\lambda v^2)$

$$\Big\{ 6m_W^2 + 3m_Z^2 - 12m_t^2 + 3m_h^2 \Big\}$$

this term is negative if

$$m_h^4 < \frac{1}{3}\left(12\, m_t^4 - 6\, m_W^4 - 3\, m_Z^4\right)$$

$$= (240 \text{ GeV})^4$$

Actually omitted finite corrections are important, and one actually finds the bound

$$m_h > 140 \text{ GeV}$$

in the Standard Model. For more details, see

M. Sher, Phys. Repts. 179, 273 (1989).