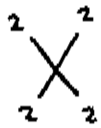
Physics 332 – Problem Set #4
Solutions

1.) a.) $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4)$

$$- \frac{2\rho}{4!}\phi_1^2\phi_2^2$$

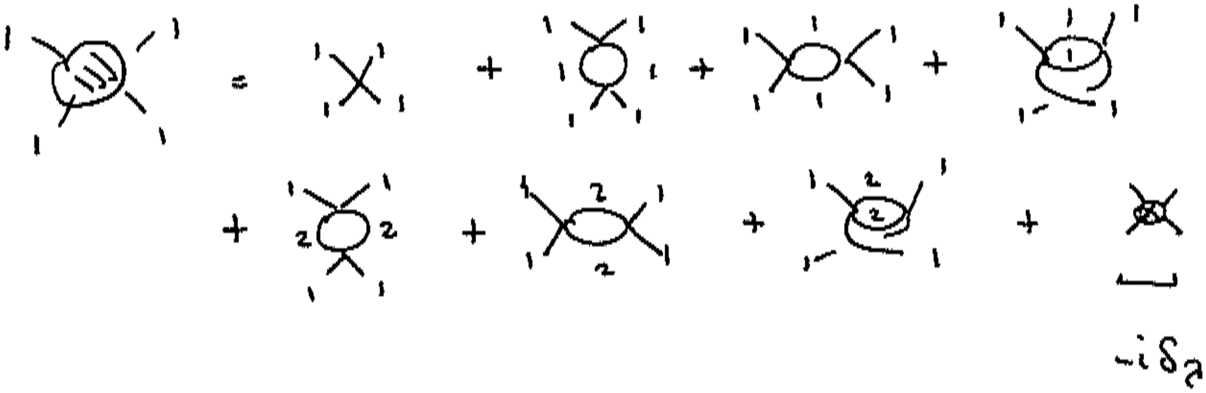
The Feynman rules are:

 =  $= -i\lambda$  $= -i\frac{1}{3}\rho$

As usual in scalar field thy, $\delta_Z = 0$ in one-loop order. So, to 1-loop order

$$\beta_\lambda = M\frac{\partial}{\partial M}(-\delta_\lambda) \qquad \beta_\rho = M\frac{\partial}{\partial M}(-\delta_\rho)$$

Let's compute:



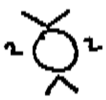
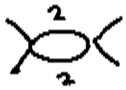
 $= \frac{1}{2}(-i\lambda)^2 \int \frac{d^dp}{(2\pi)^d} \frac{i}{p^2}\frac{i}{p^2}$

$$= \frac{1}{2}\lambda^2 \int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^2} = \frac{1}{2}\frac{i\lambda^2}{(4\pi)^{d/2}}\Gamma(2-d/2)$$

$$= \frac{i\lambda^2}{2(4\pi)^2}\lg\Lambda^2$$

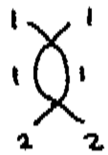
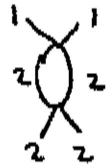
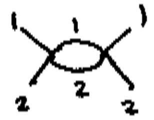
similarly

 $\cong$  $\cong \frac{i\lambda^2}{2(4\pi)^2}\lg\Lambda^2$

 $\cong$  $\cong$  $\cong \frac{i}{2}\left(\frac{\rho}{3}\right)^2\frac{1}{(4\pi)^2}\lg\Lambda^2$

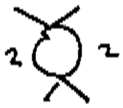
then  $= -i\delta_\lambda = (-i)\cdot 3\left(\frac{\lambda^2}{2}+\frac{\rho^2}{18}\right)\frac{1}{(4\pi)^2}\lg \Lambda^2/M^2$

so $\beta_\lambda = \frac{3\lambda^2+\rho^2/3}{(4\pi)^2}$

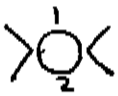
 =  +  +  +  + 

\+ 

$-i\frac{\delta\rho}{3}$

 $=$  $= \frac{1}{2}(-i\lambda)(-i\frac{\rho}{3}) \int \frac{d^dp}{(2\pi)^d} \frac{i}{p^2}\frac{i}{p^2}$

$$= \frac{1}{2}\cdot\frac{1}{3}\lambda\rho \cdot \frac{i}{(4\pi)^2} \lg \Lambda^2$$

 $=$  $= (-i\frac{\rho}{3})^2 \int \frac{d^dp}{(2\pi)^d}\frac{i}{p^2}\frac{i}{p^2}$

$$= \frac{\rho^2}{9}\frac{i}{(4\pi)^2}\lg \Lambda^2$$

so

$$-i\frac{\delta\rho}{3} = (-i)\frac{1}{(4\pi)^2}\left(\frac{\lambda\rho}{3} + \frac{2\rho^2}{9}\right)\lg \Lambda^2/M^2$$

$$\beta_\rho = \frac{2\lambda\rho + 4\rho^2/3}{(4\pi)^2}$$

again

$$\beta_\lambda = \frac{1}{(4\pi)^2}\left(3\lambda^2 + \frac{\rho^2}{3}\right)$$

$$\beta_\rho = \frac{1}{(4\pi)^2}\left(2\lambda\rho + \frac{4}{3}\rho^2\right)$$

b.) $$\frac{d}{d\lg p}\left(\frac{\rho}{\lambda}\right) = \frac{1}{\lambda}\frac{d\rho}{d\lg p} - \frac{1}{\lambda^2}\rho\,\frac{d\lambda}{d\lg p}$$

$$= \frac{1}{\lambda}\beta_\rho - \frac{\rho}{\lambda^2}\beta_\lambda$$

$$= \frac{1}{(4\pi)^2}\left\{ 2\rho + \frac{4}{3}\frac{\rho^2}{\lambda} - 3\rho - \frac{\rho^3}{3\lambda^2} \right\}$$

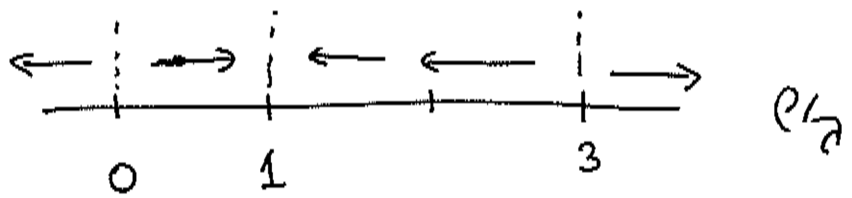
$$= \frac{1}{(4\pi)^2}\,\rho\left(-1 + \frac{4}{3}\frac{\rho}{\lambda} - \frac{1}{3}\frac{\rho^2}{\lambda^2} \right)$$

$$\frac{d}{d\lg p}\left(\frac{\rho}{\lambda}\right) = \frac{-1}{(4\pi)^2}\,\rho\left(1 - \frac{\rho}{\lambda}\right)\left(1 - \frac{1}{3}\frac{\rho}{\lambda}\right)$$

so:

as $p \to 0$

as $p \to IR$



for $0 < \rho/\lambda < 3$, $\rho/\lambda \to 1$ as $p \to 0$

Note that both ρ and λ tend to 0 as $\dfrac{1}{\lg M/p}$

c.) In $d = 4 - \varepsilon$

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2}\left(3\lambda^2 + \frac{\rho^2}{3}\right)$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2}\left(2\lambda\rho + \frac{4}{3}\rho^2\right)$$

A fixed point is a simultaneous zero of β_λ and β_ρ

Note that

$$\frac{d}{d\log p}\left(\frac{\rho}{\lambda}\right) = -\frac{1}{(4\pi)^2}\,\rho\left(1 - \frac{\rho}{\lambda}\right)\left(1 - \frac{1}{3}\frac{\rho}{\lambda}\right)$$

Just as in the previous part, the ϵ terms cancel. So we can have fixed points only at $\rho/\lambda = 0, 1, 3$

For $\underline{\rho = 0}$,

$$\beta_\rho = 0 \qquad \beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2}{(4\pi)^2}$$

$$= 0 \quad \text{at} \quad \lambda_* = \frac{(4\pi)^2\epsilon}{3}$$

For $\underline{\rho = \lambda}$.

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2}\,\frac{10}{3}\,\lambda^2$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2}\,\frac{10}{3}\,\rho^2$$

$$\text{both} = 0 \quad \text{at} \quad \lambda_* = \rho_* = (4\pi)^2 \cdot \varepsilon \cdot \frac{3}{10}$$

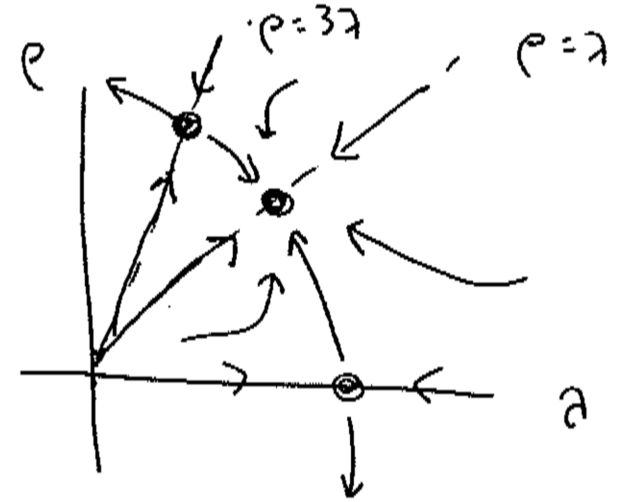
For $\underline{\rho = 3\lambda}$

$$\beta_\lambda = -\epsilon\lambda + \frac{1}{(4\pi)^2} 6\lambda^2$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{(4\pi)^2} 2\rho^2$$

both $= 0$ at $\quad \lambda_* = (4\pi)^2 \epsilon \cdot \frac{1}{6} \qquad \rho_* = (4\pi)^2 \epsilon \cdot \frac{1}{2}$

The behavior of ρ/λ determines the flow pattern:



for $\quad 0 < \frac{\rho}{\lambda} < 3$, the most stable point is

$$\rho_* = \lambda_* = \frac{3}{10} (4\pi)^2 \epsilon ,$$

an $O(2)$ - symmetric theory.

2.) The Gibbs free energy obeys

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial \rho_i} - \gamma M \frac{\partial}{\partial M} \right] G = 0$$

where M is magnetization, μ is the renormalization scale, ρ_i are coupling constants. We can specialize to the coupling constants ρ_m (relevant) and λ (slowest irrelevant) coupling. For $\lambda = \lambda_*$, the trajectory that emerges from the fixed point

$$G_*(M,t) = t^{2-\alpha} f(M t^{-\beta})$$

If $\lambda \neq \lambda_*$, then there are subleading terms proportional to $\bar{\lambda} - \lambda_*$. Let m be a mass scale; then

$$\frac{d}{d \log m/\mu} \bar{\lambda}(m) = \beta(\bar{\lambda}) = \underbrace{\beta(\lambda_*)}_{=0} + \underbrace{\frac{d\beta}{d\lambda}\Big|_{\lambda_*}}_{\equiv \omega} \cdot (\bar{\lambda} - \lambda_*) + \cdots$$

$$\frac{d}{d\, m/\mu} \bar{\lambda}(m) = \omega (\bar{\lambda} - \lambda_*) + \cdots$$

$$\bar{\lambda}(m) = \lambda_* + c\left(\frac{m}{\mu}\right)^{\omega} \qquad c \sim \lambda - \lambda_*$$

then

$$G(M,t) = G_*(M,t) + \delta G(M,t) \cdot \left(\frac{1}{\mu}\right)^{\omega}$$

Now, as t varies from 0 (T from T_c)

$$\xi^{-1} \sim (\text{mass}) \sim t^{\nu} \quad \text{so} \quad \text{mass scale} \sim t^{\nu}$$

so

$$G = G_* + \delta G \cdot (t^{\nu}/\mu)^{\omega}$$

or

$$G = t^{2-\alpha}\left[f(Mt^{-\beta}) + (\lambda-\lambda_*)\, t^{\omega\nu} g(Mt^{-\beta}) + \cdots \right]$$

Here is another, equivalent derivation. Use Peskin + Schroeder, eq. (13.31), with $\rho_i = \lambda$, M = magnetization

$$\frac{d}{d\log M}\bar{\lambda} = \frac{2}{d-2+2\gamma(\bar{\lambda})}\,\beta(\bar{\lambda})$$

$$\approx \frac{2}{d-2+2\gamma(\lambda_*)}\,\omega\,(\bar{\lambda}-\lambda_*)$$

So

$$\bar{\lambda} = \lambda_* + (\lambda-\lambda_*)\cdot(\text{const})\cdot M^{\frac{2\omega}{d-2+2\gamma_*}}$$

now $\dfrac{2}{d-2+2\gamma_*} = \dfrac{1}{2-\gamma_{\phi^2 *}}\cdot\dfrac{1}{\beta} = \nu/\beta$

so the correction term is proportional to

$$M^{\omega\nu/\beta} h(Mt^{-\beta}) = M^{\omega\nu/\beta}\,[M^{-1}t^{\beta}]^{(\nu/\beta)\cdot\omega} g(Mt^{-\beta})$$

$$= t^{\omega\nu} g(Mt^{-\beta})$$