Physics 332 – Problem Set # 3

Solutions

1.) In problem set 1, problem 1, we computed the ultraviolet divergence of the Yukawa thy. For

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi} i \not{\partial} \psi - ig\bar{\psi}\gamma^5\psi\phi$$

$$+ \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{4!}\delta_\lambda\phi^4 + \delta_\psi \bar{\psi} i\not{\partial}\psi - i\delta_g\bar{\psi}\gamma^5\psi\phi$$

we found:

$$\delta_\phi = -\frac{4g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_\lambda = -\frac{48g^4}{(4\pi)^2}\frac{1}{\epsilon} + \frac{3\lambda^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_\psi = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta_g = \frac{2g^3}{(4\pi)^2}\frac{1}{\epsilon}$$

With a momentum cutoff $\frac{2}{\epsilon} \to \log\frac{\Lambda^2}{M^2}$

so $M\frac{\partial}{\partial M}$ "$\frac{1}{\epsilon}$" $= -1$

Then, to 1-loop order

$$\beta_\lambda = M \frac{\partial}{\partial M} \left(-\delta_\lambda + 2\lambda \delta_\phi \right)$$

$$= M \frac{\partial}{\partial M} \left(-\frac{3\lambda^2}{(4\pi)^2} \frac{1}{\epsilon} + \frac{48 g^4}{(4\pi)^2} \frac{1}{\epsilon} - \frac{8 g^2 \lambda}{(4\pi)^2} \frac{1}{\epsilon} \right)$$

$$= \frac{1}{(4\pi)^2} \left(3\lambda^2 + 8\lambda g^2 - 48 g^4 \right)$$

$$\beta_g = M \frac{\partial}{\partial M} \left(-\delta_g + g\,\delta_\psi + \tfrac{1}{2} g\,\delta_\phi \right)$$

$$= M \frac{\partial}{\partial M} \left(-\frac{2g^3}{(4\pi)^2} \frac{1}{\epsilon} + \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} - \frac{2g^2}{(4\pi)^2} \frac{1}{\epsilon} \right)$$

$$= \frac{3g^3}{(4\pi)^2}$$

again:

$$\beta_g = \frac{3g^3}{(4\pi)^2} \qquad \beta_\lambda = \frac{1}{(4\pi)^2} \left(3\lambda^2 + 8\lambda g^2 - 48 g^4 \right)$$

β_g is always positive. But β_λ can be zero, at

$$0 = 3\lambda^2 + 8\lambda g^2 - 48 g^4$$

$$= (\lambda/g^2)^2 + 2 \cdot \tfrac{4}{3} (\lambda/g^2) - 16$$

$$= \left(\lambda/g^2 + \tfrac{4}{3} \right)^2 - 16 - \tfrac{16}{9}$$

or $\frac{\lambda}{g^2} = -\frac{4}{3} \pm \frac{4}{3}\sqrt{10}$

so

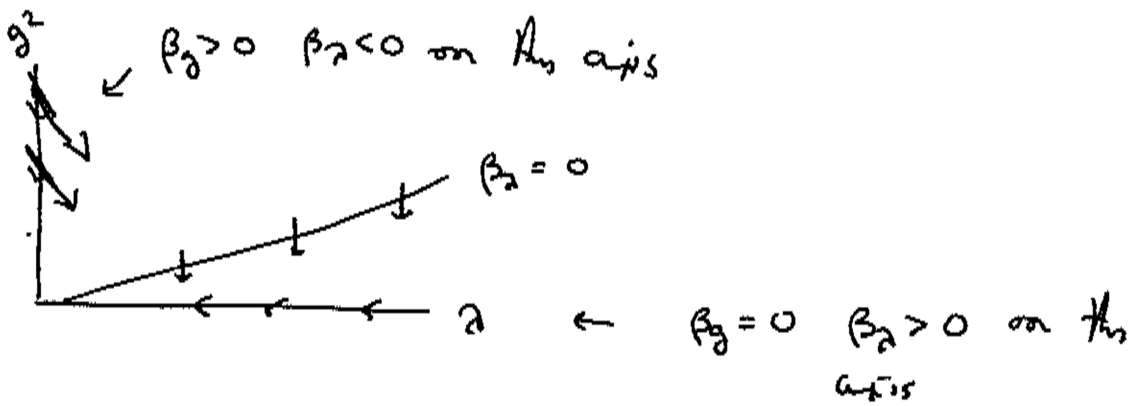
$$\beta_\lambda = 0 \quad \text{on} \quad \lambda = g^2 \cdot \tfrac{4}{3}(\sqrt{10}-1)$$

$$\beta_\lambda > 0 \text{ for } \lambda > \text{ , } < 0 \text{ for } \lambda <$$

then

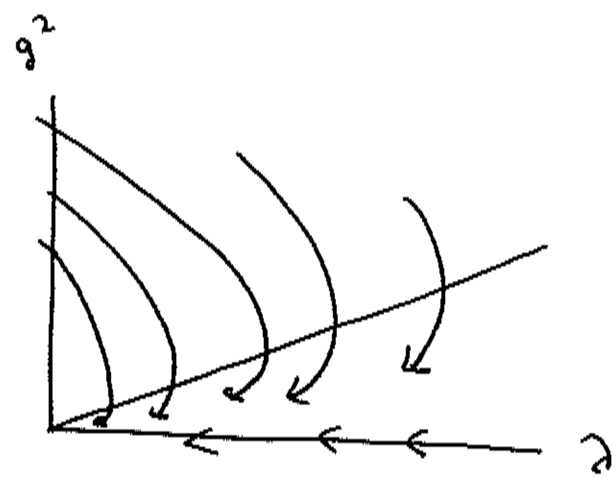
flows to IR

or $M \to 0$



the flow diagram for $\frac{d}{d \log M} \begin{pmatrix} \lambda(M) \\ g^2(M) \end{pmatrix} = \begin{pmatrix} \beta_\lambda \\ 2g\,\beta_g \end{pmatrix}$

is

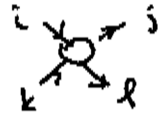


with $g^2, \lambda \to 0$ as $M \to 0$

2.) In the Gross-Neveu model

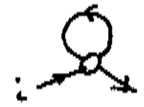
$$\mathcal{L} = \bar\Psi_i i\partial\!\!\!/ \Psi_i + \tfrac{1}{2} g^2 (\bar\Psi_i \Psi_i)^2 \quad \text{in } d=2$$

the Feynman rule is

 $= +ig^2(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk}) = -ig^2($  $-$  $)$

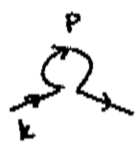
$$\bar\Psi\,\Psi\,\bar\Psi\,\Psi \qquad\qquad \bar\Psi\,\Psi\,\bar\Psi\,\Psi = (-1)$$

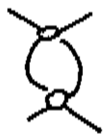
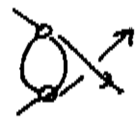
fermion minus sign

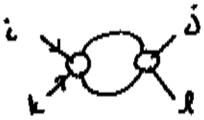
 $=$  $= -ig^2[\bar\Psi_i\Psi_i\underbrace{\bar\Psi_j\Psi_j} + \bar\Psi_i\Psi_i\bar\Psi_j\Psi_j]$

$$\bar\Psi_j\Psi_j = \int\frac{d^2p}{(2\pi)^2}(-1)\,\mathrm{tr}\,\frac{ip\!\!\!/}{p^2} = 0 \text{ in the}$$

massless theory

$= \delta_{ij}\int\frac{d^2p}{(2\pi)^2}(+ig^2)\frac{ip\!\!\!/}{p^2} = 0$ 

 $=$  $+$  $+$ 

 $=$  $-$  $-$  $+$ 

$$= (+ig)^2(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj})\cdot 2\cdot\tfrac{1}{2}\cdot\int\frac{d^2p}{(2\pi)^2}\,\frac{ip\!\!\!/}{p^2}\,\frac{ip\!\!\!/}{p^2}$$

Note that this is a term of $O(1)$. The other diagrams have terms of $O(N)$

(sum) = 

$$= (+ig^2)^2\,(\delta^{ij}\delta^{kl}-\delta^{il}\delta^{kj})\,(-1)\cdot N\cdot\int\frac{d^2p}{(2\pi)^2}\,\operatorname{tr}\left[\frac{i\not p}{p^2}\,\frac{i\not p}{p^2}\right]$$

$$= +g^4N\,(\delta^{ij}\delta^{kl}-\delta^{il}\delta^{kj})\int\frac{d^2p}{(2\pi)^2}\cdot 2\cdot\frac{1}{p^2}$$ $\operatorname{tr}\underline{1=2}$ in $d=2$

$$= +g^4N\,(\delta^{ij}\delta^{kl}-\delta^{il}\delta^{kj})\;2\cdot\frac{-i}{(4\pi)^{d/2}}\,\Gamma(1-d/2)$$

$$= i\,g^4N\,(\delta^{ij}\delta^{kl}-\delta^{il}\delta^{kj})\;\frac{2}{4\pi}\,\log\frac{\Lambda^2}{M^2}$$

so

$$\delta_{g^2} = -\frac{1}{2\pi}\,g^4N\,\log\Lambda^2/M^2$$

$$\beta_{g^2} = \frac{dg^2}{d\log M} = -\frac{g^4N}{\pi} + O(1)$$

$$\beta_g = \frac{dg}{d\log M} = -\frac{g^3N}{2\pi} + O(1)$$

Integrating this equation, we find

$$g^2(M) = \frac{g_0^2}{1+\frac{g_0^2N}{\pi}\log M^2/M_0^2} \longrightarrow 0 \text{ as } M\to\infty$$

In the previous problem set, we found

$$m_\psi = M e^{-\pi/g^2 N}$$

Since m_ψ is physical (measurable), its normalization should be unambiguous. If we change M (defined by convention), but change g^2 accordingly, the value of m_ψ should be unchanged.

Check this:

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} \right] m_\psi$$

$$= m_\psi + \beta(g) \cdot \frac{2\pi}{g^3 N} \cdot m_\psi = 0 \quad \checkmark$$

3.) In problem set # 1, problem 2 we found in ϕ^4 theory

$$= -i \frac{\lambda^2}{12} \frac{p^2}{(4\pi)^4} \Big(\underbrace{-\frac{1}{\epsilon} + \log p^2}_{\left(-\frac{\Gamma(4-d)}{[p^2]^{4-d}}\right)} + \cdots \Big)$$

$$= -i \frac{\lambda^2}{12} \frac{p^2}{(4\pi)^4} \left(-\log \frac{\Lambda^2}{p^2} + \cdots \right)$$

then

$$= i \delta_Z p^2 \qquad \delta_Z = -\frac{\lambda^2}{12} \frac{1}{(4\pi)^4} \log \frac{\Lambda^2}{M^2}$$

this is the first correction to Z, so

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M} \log Z = \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z$$

$$= \frac{\lambda^2}{12} \frac{1}{(4\pi)^4}$$

Now repeat for the O(N) linear sigma model.

 $= -i\lambda \quad \Rightarrow$  $= -2i\lambda \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$

then

ϕ^4 thy  $= \frac{(-i\lambda)^2}{3!} \cdot \int \frac{d^dq\, d^dr}{(2\pi)^{2d}} \cdots$

O(N) linear s.m.  $= \frac{(-2i\lambda)^2}{3!} \left(\delta^{ia}\delta^{bc} + \delta^{ib}\delta^{ac} + \delta^{ic}\delta^{ab}\right)$

$$\cdot \left(\delta^{ja}\delta^{bc} + \delta^{jb}\delta^{ac} + \delta^{jc}\delta^{ab}\right) \cdot \int \frac{d^dq\, d^dr}{(2\pi)^{2d}} \cdots$$

that is

$$\frac{(-i\lambda)^2}{3!} \rightarrow \frac{(-i\lambda)^2}{3!} \cdot 4 \cdot \left(\delta^{ia}\delta^{bc} + \delta^{ib}\delta^{ac} + \delta^{ic}\delta^{ab}\right)\left(\delta^{ja}\delta^{bc} + \delta^{jb}\delta^{ac} + \delta^{jc}\delta^{ab}\right)$$

$$= \frac{(-i\lambda)^2}{3!} \cdot 4 \cdot \left(3 \cdot N \delta^{ij} + 6 \delta^{ij}\right)$$

using $\delta^{ab}\delta^{ab} = N$

$$= \frac{(-i\lambda)^2}{3!} \cdot 12 \cdot (N+2)\,\delta^{ij}$$

so

$$= \quad -i\,(N+2)\,\lambda^2\,\frac{p^2}{(4\pi)^4}\left(-\log \Lambda^2/p^2\right)$$

$$\delta_Z = -\frac{\lambda^2 (N+2)}{(4\pi)^4}\log \Lambda^2/M^2$$

$$\gamma = \frac{(N+2)}{(4\pi)^4}\,\lambda^2$$