Physics 332 - Problem Set #2

Solutions

1.) a.) We discussed the formula for $\langle e^{i\phi} \rangle$ in Physics 331, problem set #2. But here is the derivation again: Work with a Euclidean field, with eq. of motion

$$\Delta_x \phi = 0 \qquad \text{eg.} \qquad \Delta_x = (-\partial^2 + m)$$

Let $D(x)$ be the Green's funct for Δ

$$\Delta_x D(x-y) = \delta^{(4)}(x-y)$$

then

$$Z = \int \mathcal{D}\phi \; e^{-\int dx \, \frac{1}{2} \phi \Delta_x \phi}$$

$$\langle \phi(x) \phi(y) \rangle = \frac{\int \mathcal{D}\phi \; e^{-\int dx \, \frac{1}{2} \phi \Delta_x \phi} \; \phi(x) \phi(y)}{\int \mathcal{D}\phi \; e^{-\int dx \, \frac{1}{2} \phi \Delta_x \phi}}$$

$$= D(x-y)$$

To derive this, add a source J :

$$Z[J] = \int \mathcal{D}\phi \; e^{-\int dx \, \frac{1}{2}\phi \Delta_x \phi + J\phi}$$

and evaluate

$$Z[J] = \int \mathcal{D}\phi \; e^{-\int \frac{1}{2}(\phi + \Delta^{-1}J)\,\Delta\,(\phi + \Delta^{-1}J) - J\Delta^{-1}J}$$

let $\phi' = \phi + \Delta^{-1}J$ $\qquad \Delta^{-1}J = \int dz \; D(x-z)\,J(z)$

$$Z[J] = (\text{const})\cdot \exp\left[\frac{1}{2}\int dy\,dz\; J(y)\,D(y-z)\,J(z)\right]$$

for $J_0(y) = i[\delta(y-x) - \delta(y)]$

$$\langle e^{i[\phi(x) - \phi(0)]}\rangle = \frac{Z[J_0]}{Z[0]}$$

$$= \exp\left[\frac{(i)^2}{2}\int dy\,dz \; [\delta(y-x) - \delta(y)]\,D(y-z)\,[\delta(z-x) - \delta(z)]\right]$$

$$= \exp\left[-D(0) + D(x)\right] \quad \text{as required}$$

b.) Since ϕ has positive mass dimension $(\text{mass})^{d-2}$

$(\vec{\nabla}\phi)$ has dimension > 1

$(\vec{\nabla}\phi)^4$ has dimension > 4, and so is

nonrenormalizable in $d > 2$

so take $\mathcal{R} = \frac{1}{2} \rho (\vec{\nabla}\phi)^2$

$$\langle \phi(x)\, \phi(0) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\rho} \frac{1}{k^2} e^{ik\cdot x}$$

the right-hand side is the Green's function of the Laplace operator. If $(-\nabla^2)\, G(x-y) = \delta(x-y)$

$$\vec{E} = -\vec{\nabla} G$$

satisfies $\int d\hat{n}\cdot\vec{E} = 1$

so

$$\vec{E} = \frac{\hat{x}}{x^{d-1}} \cdot \frac{1}{A(d)}$$

$A(d)$ = area of unit sphere

$$G = \frac{1}{|x|^{d-2}} \frac{1}{A(d)} \frac{1}{d-2}$$

so

$$G = \begin{cases} \underline{d=3} & \dfrac{1}{4\pi |x|} \\ d=2 & -\dfrac{1}{2\pi} \log |x| \\ d=1 & -\dfrac{1}{2} |x| \end{cases}$$

$\langle \phi(x)\, \phi(0) \rangle = \frac{1}{\rho} \times$ (the above) then Let $s = A e^{i\phi(x)}$

d=1

$$\langle s(x)\, s^*(0)\rangle = A^2\, e^{\langle\phi(x)\phi(0)\rangle - \langle\phi^2(0)\rangle}$$

$$= A\, e^{-\frac{1}{2\rho}|x|}$$

$\langle s(x)\, s^*(0)\rangle \longrightarrow 0$ as $|x| \to \infty$, no spontaneous symmetry breaking.

d=2

$$\langle s(x)\, s^*(0)\rangle = A^2\, e^{-\frac{1}{2\pi\rho}\log|x| - (\text{const})}$$

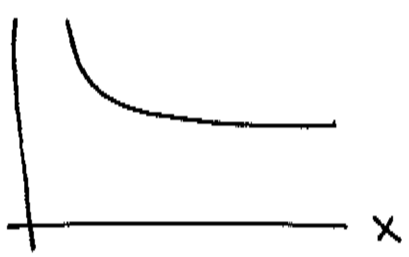
$$\sim \frac{B}{|x|^{\frac{1}{2\pi\rho}}}$$

again $\langle s(x)\, s^*(0)\rangle \longrightarrow 0$ no spontaneous symmetry breaking.

d=3

$$\langle s(x)\, s^*(0)\rangle = A^2\, e^{\frac{1}{4\pi|x|\cdot\rho} - (\text{const})}$$

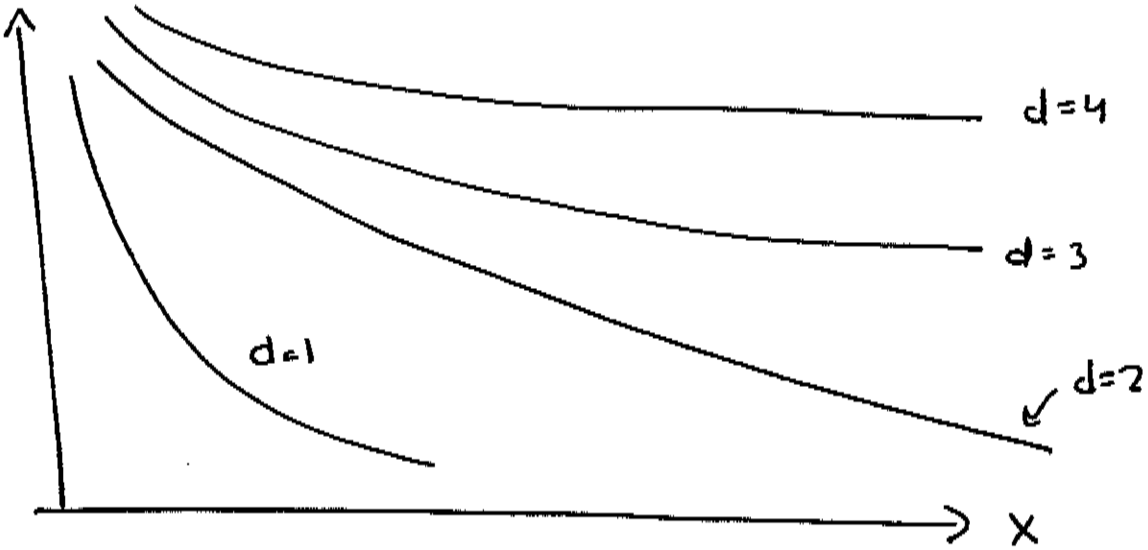
$$= B\, e^{\frac{1}{4\pi\rho}\frac{1}{x}}$$



$\longrightarrow$ const as $x \to \infty$

more generally,

$\langle s(x)\, s^*(0) \rangle$



d=2 is the boundary dimension at which fluctuations of spin waves destroy spontaneous symmetry breaking.

2.) a.) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2$
$+ \bar{\Psi} i \not{\partial} \psi - g\,\bar{\Psi}(\phi^1 + i\gamma^5\phi^2)\psi$ $\quad i = 1,2$

under $\phi^1 \to \cos\alpha\,\phi^1 - \sin\alpha\,\phi^2 \qquad \phi^2 \to \sin\alpha\,\phi^1 + \cos\alpha\,\phi^2$

$(\phi^1)^2 + (\phi^2)^2 \longrightarrow (\phi^1)^2 + (\phi^2)^2$ so the first line of $\mathcal{L}$ is invariant

also, since $(\gamma^5)^2 = 1$

$$(\phi^1 + i\gamma^5\phi^2) \to (\cos\alpha\,\phi^1 - \sin\alpha\,\phi^2) + i\gamma^5(\sin\alpha\,\phi^1 + \cos\alpha\,\phi^2)$$

$$= (\cos\alpha + i\gamma^5\sin\alpha)\phi^1 + i\gamma^5(\cos\alpha + i\sin\alpha\,\gamma^5)\phi^2$$

$$= e^{i\alpha\gamma^5}(\phi^1 + i\gamma^5\phi^2)$$

so if also $\quad \psi \to e^{-i\alpha\gamma^5/2}\psi$

$$\bar{\Psi} \to \psi^+ e^{+i\alpha\gamma^5/2}\gamma^0$$

$$= \bar{\Psi}\, e^{-i\alpha\gamma^5/2}$$

then $\quad \bar{\Psi} i\not{\partial}\psi \to \bar{\Psi} i \not{\partial}\psi$

$$\bar{\Psi}(\phi^1 + i\gamma^5\phi^2)\psi \to \bar{\Psi}\, e^{-i\alpha\gamma^5/2}\, e^{i\alpha\gamma^5}(\phi^1 + i\gamma^5\phi^2) e^{-i\alpha\gamma^5/2}\psi$$

$$= \bar{\Psi}(\phi^1 + i\gamma^5\phi^2)\psi$$

so the entire Lagrangian is invariant to the U(1) symmetry!

The equation of motion for ϕ^i is

$$0 = -\partial^2\phi^i + \mu^2\phi^i - \lambda\,\phi^i(\phi^i)^2 - g\begin{cases}\bar\psi\psi & i=1\\ i\bar\psi\gamma^5\psi & i=2\end{cases}$$

If $\psi, \bar\psi = 0$ in the vacuum configuration, ϕ^i = const, we can solve

$$\mu^2 - \lambda(\phi^i)^2 = 0 \quad\text{or}\quad v^2 = \langle|\phi^i|^2\rangle = \frac{\mu^2}{\lambda}$$

It is most convenient to chose coordinates s.t.

$$\langle\phi^1\rangle = v \qquad \langle\phi^2\rangle = 0$$

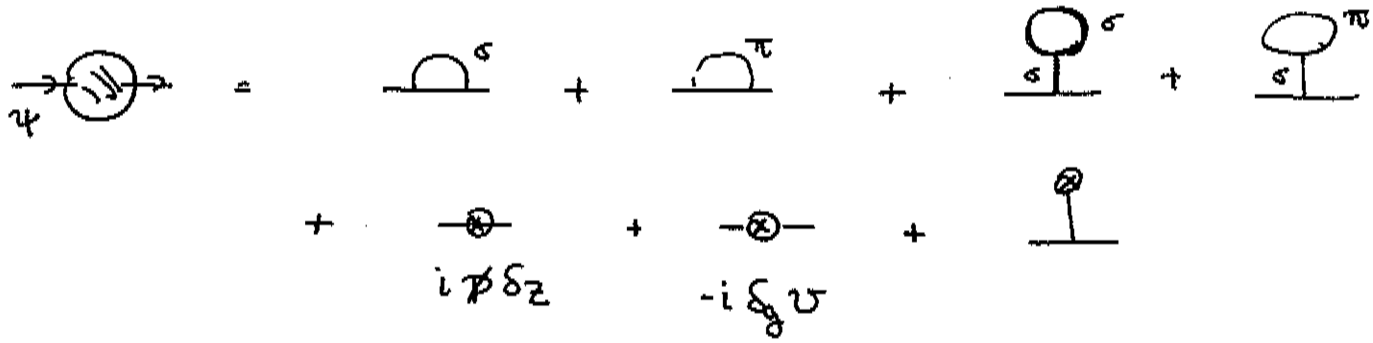
Any other solution can be rotated into this one by the U(1) symmetry.

b.) Set $\phi^1 = v + \sigma \quad \phi^2 = \pi$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\lambda\sigma)^2 + \tfrac{1}{2}(\partial_\lambda\pi)^2 - \mu^2\sigma^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2$$
$$- \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2\pi^2 - \frac{\lambda}{4}\pi^4$$
$$+ \bar\psi(i\not\partial - gv)\psi - g\sigma\bar\psi\psi - ig\bar\psi\gamma^5\psi\pi$$

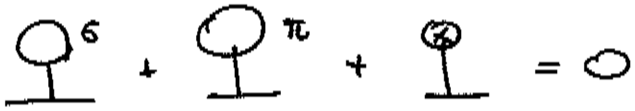
Indeed $m_f = gv$.

c.) The correction to the fermion self-energy is



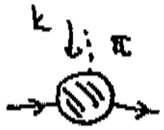
the last diagram is $-ig\left(\frac{+i}{-m_\sigma^2}\right)\cdot$ [counterterm $\propto \sigma$]

Fortunately, if we insist that $\langle\phi\rangle = v$ as a renormalization condition then



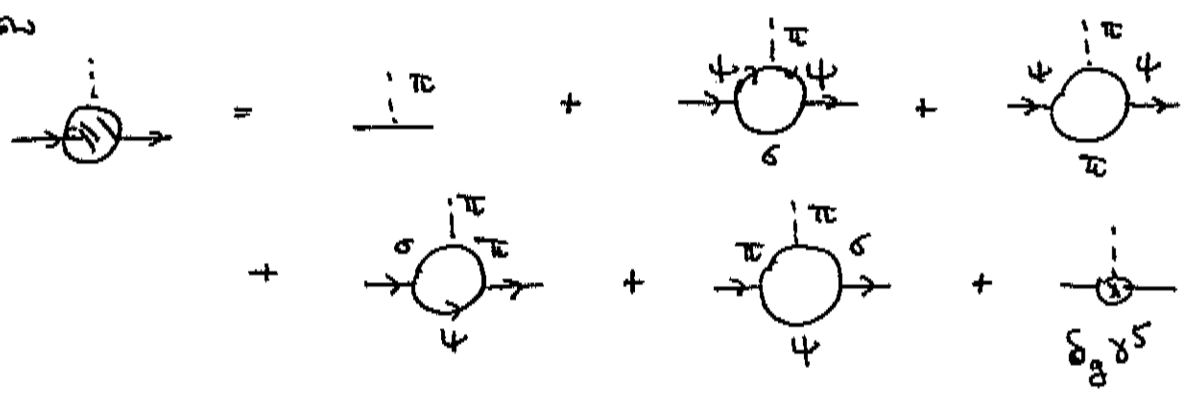
δ_Z is determined by the condition $\frac{d}{d\not{p}}$  $\Big|_{\not{p}=m_f} = 0$

δ_g is determined [according to the problem set] by the condition



$= g\gamma^5$ at $k=0$ ψ on shell

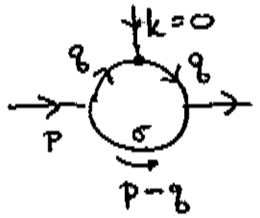
Now



This assumes that $Z_\psi = Z_\pi = 1$; otherwise there are additional factors from $\sqrt{Z}$'s. The condition $Z_\pi = 1$ assumes we have imposed the additional renormalization condition

$$\frac{d}{dp^2}\left[\ \pi\ \right]\Big|_{p^2=0} = 0$$

Anyway, with this understood, let's compute δ_g:



$$= (-ig)^2 g \int \frac{d^dq}{(2\pi)^d} \frac{i(\not q+m_f)\,\gamma^5\, i(\not q+m_f)}{(q^2-m_f^2)^2}\,\frac{i}{(p-q)^2-m_\sigma^2}$$

$$= ig^3\gamma^5 \int \frac{d^dq}{(2\pi)^d} \frac{(-\not q+m_f)(\not q+m_f)}{(q^2-m_f^2)^2\ \ (p-q)^2-m_\sigma^2}$$

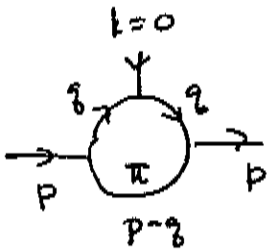
$(\not q-m_f)(\not q+m_f) = q^2-m_f^2$

$$= -ig^3\gamma^5 \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2-m_f^2)((p-q)^2-m_\sigma^2)}$$

$$= -ig^3\gamma^5 \int_0^1 dx\ \frac{i}{(4\pi)^{d/2}}\,\Gamma(2-\tfrac{d}{2})\,\frac{1}{\left[x m_\sigma^2+(1-x)m_f^2 - x(1-x)p^2\right]^{2-\frac{d}{2}}}$$

and set $p^2 = m_f^2$

$$= -ig^3\gamma^5\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-\tfrac{d}{2}) \int_0^1 dx\ \frac{1}{\left[x m_\sigma^2 + (1-x)^2 m_f^2\right]^{2-\frac{d}{2}}}$$

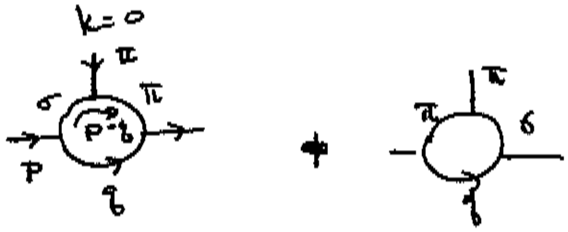


$$= g^3 \int \frac{d^dq}{(2\pi)^d}\ \gamma^5\ \frac{i(\not q+m_f)\,\gamma^5\, i(\not q+m_f)}{(q^2-m_f^2)^2}\,\gamma^5\,\frac{i}{(p-q)^2}$$

$$= -ig^3 \int \frac{d^dq}{(2\pi)^d}\ \frac{(\not q+m_f)(-\not q+m_f)}{(q^2-m_f^2)(\ (p-q)^2)} \quad \leftarrow\ m_\pi = 0$$

$$= +ig^3 \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2-m_f^2)((p-q)^2)}$$

$$= ig^3 \gamma 5 \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \int_0^1 dx \frac{1}{[(1-x)^2 m_f^2]^{2-d_2}}$$



$$= (-ig)(g)(-2i\lambda v)$$

$$\cdot \int \frac{d^dq}{(2\pi)^d} \left[\gamma 5 \frac{i(\not q+m_f)}{q^2-m_f^2} + \frac{i(\not q+m_f)}{q^2-m_f^2} \gamma 5\right] \frac{i}{(p-q)^2-m_\sigma^2} \frac{i}{[(p-q)^2]}$$

$$= 2\cdot 2i\lambda v g^2 \gamma^5 m_f \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2-m_f^2} \frac{1}{(p-q)^2-m_\sigma^2} \frac{1}{(p-q)^2}$$

$$= 4i\lambda v g^2 \gamma^5 m_f \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2-m_f^2} \int_0^1 dy \frac{1}{[(p-q)^2 - y m_\sigma^2]^2}$$

$$= 4i\lambda v g^2 \gamma^5 m_f \int \frac{d^dq}{(2\pi)^d} \int_0^1 dx \; x \cdot 2 \int_0^1 dy \frac{1}{\left[q^2 + x(1-x)p^2 - xym_\sigma^2 - (1-x)m_f^2\right]^3}$$

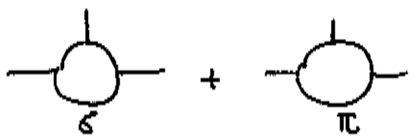
$$= 4i\lambda v g^2 \gamma^5 m_f \frac{-i}{(4\pi)^{d_2}} \frac{\Gamma(3-d_2)}{\Gamma(3)} \int_0^1 dx \int_0^1 dy \; 2x \frac{1}{[xym_\sigma^2 + (1-x)^2 m_f^2]^{3-d_2}}$$

we can set $d=4$ and do the y integral

$$= 4\lambda v g^2 \gamma^5 m_f \frac{1}{(4\pi)^2} \int dx \int dy \frac{x}{[x\, y m_\sigma^2 + (1-x)^2 m_f^2]}$$

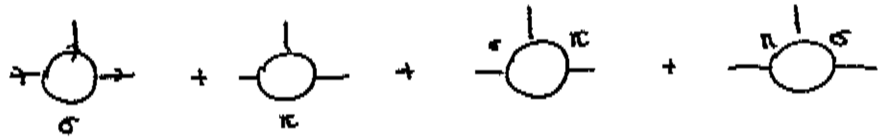
$$= 4\lambda v g^2 \gamma^5 \frac{m_f}{m_\sigma^2} \frac{1}{(4\pi)^2} \int_0^1 dx \log\left[\frac{xm_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]$$

similarly the divergences of the first two diagrams cancel [this is something of an accident] and we find



$$= -g^3 \gamma^5 \frac{1}{(4\pi)^2} \int_0^1 dx \quad \log\left[\frac{x\, m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]$$

$\simeq$ all



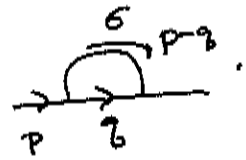
$$= (-\gamma^5) \frac{1}{(4\pi)^2} \left[g^3 - 4\lambda v \frac{g^2 \cdot g v}{m_\sigma^2} \right] \int_0^1 dx \quad \log\left(\frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right)$$

$$m_\sigma^2 = 2\mu^2 = 2\lambda v^2$$

$$= +\gamma^5 \frac{1}{(4\pi)^2} g^3 \left[\int_0^1 dx \quad \log \frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]$$

$$= -\gamma^5 \delta g$$

Now turn to:



$$= (-ig)^2 \int \frac{d^d q}{(2\pi)^d} \frac{i(\not{q}+m_f)}{(q^2-m_f^2)} \quad \frac{i}{(p-q)^2 - m_\sigma^2}$$

$$\qquad\qquad (1-x) \qquad\qquad\qquad x$$

Feynman parameters $\longrightarrow$

$$\text{Denom} = q^2 - 2xp + xp^2 - x m_\sigma^2 + (1-x) m_f^2$$
$$= q'^2 + x(1-x)p^2 - (x m_\sigma^2 + (1-x) m_f^2)$$

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$$q = \mathbb{q} + xp$$

$$= g^2 \int \frac{d^d \mathbb{q}}{(2\pi)^d} \int_0^1 dx \frac{\not{\mathbb{q}} + x\not{p} + m_f}{\left[\mathbb{q}^2 + x(1-x) p^2 - (x m_\sigma^2 + (1-x) m_f^2) \right]^2}$$

$$= g^2 \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) \int_0^1 dx \frac{x\not{p} + m_f}{\left[x m_\sigma^2 + (1-x) m_f^2 - x(1-x) p^2 \right]^{2-d/2}}$$

$$= g^2 \int \frac{d^d q}{(2\pi)^d} \gamma_5 \frac{i(\not{q} + m_f)}{q^2 - m_f^2} \gamma_5 \frac{i}{(p-q)^2}$$

$$= g^2 \int \frac{d^d \mathbb{q}}{(2\pi)^d} \int_0^1 dx \frac{(x\not{p} - m_f)}{\left[\mathbb{q}^2 + x(1-x) p^2 - (1-x) m_f^2 \right]^2}$$

$$= g^2 \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) \int_0^1 dx \frac{x\not{p} - m_f}{\left[(1-x) m_f^2 - x(1-x) p^2 \right]^{2-d/2}}$$

in all

$+$

$$= -i \frac{g^2}{(4\pi)^2} \Big\{ -\frac{2}{\epsilon} \not{p}$$

$$+ \int_0^1 dx \left[x\not{p} \left\{ \log \frac{x m_\sigma^2 + (1-x) m_f^2 - x(1-x) p^2}{M^2} + \log \frac{(1-x) m_f^2 - x(1-x) p^2}{M^2} \right\} \right.$$

$$\left. + m_f \log \left(\frac{x m_\sigma^2 + (1-x) m_f^2 - x(1-x) p^2}{(1-x) m_f^2 - x(1-x) p^2} \right) \right] \Big\}$$

write this as

$$= i \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \not{p} \quad - i \frac{g^2}{(4\pi)^2} \hat{\Sigma}(\not{p})$$

then, in all, the inverse fermion propagator is

$$(\not p - m_f) + \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\not p - \frac{g^2}{(4\pi)^2}\hat\Sigma(\not p) + \not p\,\delta_Z - \delta_g v$$

where δ_g is given on p. 6. The fermion mass is given by

$$0 = \not p - m_f + \left(\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} + \delta_Z\right)\not p - \frac{g^2}{(4\pi)^2}\hat\Sigma(p) - \delta_g v$$

where δ_Z is determined by

$$0 = \left(\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon} + \delta_Z\right) - \frac{g^2}{(4\pi)^2}\frac{d\hat\Sigma}{d\not p}\bigg|_{\not p = m_f}$$

This removes the last ∞ in the expression for M_f!

so the solution to the equation for m_f is

$$\not p = m_f - \not p\left(\frac{g^2}{(4\pi)^2}\frac{d\hat\Sigma}{d\not p}\right)\bigg|_{\not p = m_f} + \frac{g^2}{(4\pi)^2}\hat\Sigma(\not p) + \delta_g v$$

is the exact fermion mass, through order g^2, is

$$M_f = m_f + \frac{g^2}{(4\pi)^2}\hat\Sigma(m_f) + \delta_g v - \frac{g^2}{(4\pi)^2} m_f \frac{d\Sigma}{d\not p}\bigg|_{m_f}$$

$$= m_f + \frac{g^2}{(4\pi)^2}\int_0^1 dx\left[m_f \cancel{\lg\left[\frac{x m_\sigma^2 + (1-x)m_f^2 - x(1-x)m_f^2}{(1-x)m_f^2 - x(1-x)m_f^2}\right]}\right]$$

$$-2x(1-x)m_f^2\left(\frac{1}{x m_\sigma^2 + (1-x)m_f^2 - x(1-x)m_f^2} - \frac{1}{(1-x)m_f^2 - x(1-x)m_f^2}\right)$$

$m_f = g v$

$$- x m_f \cdot 2x(1-x)m_f\left(\frac{1}{x m_\sigma^2 + (1-x)m_f^2 - x(1-x)m_f^2} + \frac{1}{(\qquad)}\right)$$

$$-\frac{g^3 v}{(4\pi)^2}\int_0^1 dx\, \cancel{\lg\left[\frac{x m_\sigma^2 + (1-x)^2 m_f^2}{(1-x)^2 m_f^2}\right]}$$

finally

$$M_f - gv = -\frac{g^2}{(4\pi)^2}\int_0^1 dx \Big\{ 2x(1-x)\, m_f^2 (1+x) \frac{1}{x m_\sigma^2 + (1-x)^2 m_f^2}$$

$$- 2x(1-x)^2 m_f^2 \frac{1}{(1-x)^2 m_f^2} \Big\}$$

or

$$M_f - gv = -\frac{g^2}{(4\pi)^2}\left(\int_0^1 dx \frac{2x(1-x^2)\, m_f^2}{x m_\sigma^2 + (1-x)^2 m_f^2} - 1\right)$$

2.) a) $\mathcal{L} = \bar{\Psi}_i i\not\partial \Psi_i + \frac{1}{2} g^2 (\bar{\Psi}_i \Psi_i)^2$

$\gamma^5 = \gamma^0\gamma^1 \qquad \{\gamma^5, \gamma^\mu\} = 0 \qquad (\gamma^5)^2 = 1$

under $\quad \Psi_i \to \gamma^5 \Psi_i \qquad \bar{\Psi}_i \to \bar{\Psi}_i(-\gamma^5)$

$\bar{\Psi}_i i\not\partial \Psi_i \to +\bar{\Psi}_i i\not\partial \Psi_i$

$\bar{\Psi}_i \Psi_i \to -\bar{\Psi}_i \Psi_i$

so $\mathcal{L} \to \mathcal{L}$ by $\quad \delta\mathcal{L} = -m\bar{\Psi}_i\Psi_i \to -\delta\mathcal{L}$

the mass term for Ψ_i does not respect the discrete symmetry b.) In 2-D, $\partial_\mu \sim (\text{mass})^1$

$$\bar{\Psi} \not{\partial} \psi \sim (\text{mass})^d = (\text{mass})^2$$

so $\psi \sim (\text{mass})^{1/2}$

then $\bar{\Psi}\psi \sim (\text{mass})$

$$(\bar{\Psi}\psi)^2 \sim (\text{mass})^2 = (\text{mass})^d$$

since $\mathcal{L} \sim (\text{mass})^d$ [$\int d^4x\, \mathcal{L}$ is dim-less]

g must be dimensionless $\rightarrow$ the thy is renormalizable.

c.) Since $\int \mathcal{D}\sigma \; e^{i\int(-\frac{1}{2g^2}\sigma^2 - \sigma\bar{\Psi}_i\psi_i)\,d^2x}$

$$= \int \mathcal{D}\sigma \; e^{i\int d^2x\{-\frac{1}{2g^2}(\sigma + g^2\bar{\Psi}_i\psi_i)^2 + \frac{g^2}{2}(\bar{\Psi}_i\psi_i)^2}$$

$$= (\text{const}) \; e^{i\int d^4x\, \frac{g^2}{2}(\bar{\Psi}_i\psi_i)^2}$$

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\Psi}\mathcal{D}\sigma \; e^{i\int d^2x(\bar{\Psi}_i i\not{\partial}\psi_i - \sigma\bar{\Psi}_i\psi_i - \frac{1}{2g^2}\sigma^2)}$$

$$= \int \mathcal{D}\psi\mathcal{D}\bar{\Psi} \; e^{i\int d^2x(\bar{\Psi}_i i\not{\partial}\psi_i + \frac{g^2}{2}(\bar{\Psi}_i\psi_i)^2)}$$

Note that $\sigma \rightarrow -\sigma$ to preserve the discrete symmetry d.) Anticipating that we will need to renormalize the coupling g^2, write

$$\mathcal{L} = \bar{\Psi}_i(i\not\partial - \sigma)\Psi_i - \frac{1}{2g^2}\sigma^2 - \underbrace{\delta_g}_{\text{counterterm}}\sigma^2$$

Integrating over $\Psi_i \bar{\Psi}_i$ gives

$$Z = \int \mathcal{D}\sigma\, [\det(i\not\partial - \sigma)]^N \, e^{i\int\left(-\frac{1}{2g^2}\sigma^2 - \delta_g\sigma^2\right)d^2x}$$

Now evaluate the determinant for $\sigma(x) \approx \sigma = \underline{\text{const}}$. In a single Fourier mode

$$\det(i\not\partial - \sigma) = \det(\not k - \sigma)$$

$$= \det(\not k - \sigma) \qquad \text{for } k = (k, 0)$$

$$= \det\begin{pmatrix} -\sigma & -ik \\ ik & -\sigma \end{pmatrix} \qquad \sigma^0 = \sigma^2$$

$$= \sigma^2 - k^2 = \sigma^2 + k_E^2$$

then

$$\log\det(i\not\partial - \sigma) = \sum_k \log(\sigma^2 + k_E^2)$$

$$= (VT)\int\frac{d^dk}{(2\pi)^d}\log(\sigma^2 + k^2)$$

$$= (VT)\,(-i)\,\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\,(\sigma^2)^{d/2}$$

Near $d=2$ $\quad \Gamma(-d/2) = \frac{1}{1-d/2}\Gamma(1-d/2)$

$d = 2-\epsilon$ $\quad = \frac{1}{-1+\epsilon/2}\Gamma(\epsilon/2)$

$$= -(1+\epsilon/2)\left(\frac{2}{\epsilon} - \gamma + \cdots\right)$$

$$\log\det(i\not{\partial} - \sigma) = (VT)\,\frac{i}{4\pi}\sigma^2\left(\frac{2}{\epsilon} + 1 + \log\frac{M^2}{\sigma^2}\right)$$

where M^2 is the $\overline{MS}$ mass.

$$Z = \int \mathcal{D}\sigma \; \exp\left[i\int d^2x \left\{ -\frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\frac{2}{\epsilon} + 1 + \log\frac{M^2}{\sigma^2}\right) - \delta_g \sigma^2 \right\} \right]$$

with the renormalization, cancel the $\frac{2}{\epsilon}$ term

We then find an effective Lagrangian for σ

$$\mathcal{L}_{eff} = -\frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{M^2}{\sigma^2} + 1\right)$$

e.) $$\frac{\delta \mathcal{L}}{\delta\sigma} = 0 \Rightarrow -\frac{1}{2g^2} + \frac{N}{4\pi}\left(\log\frac{M^2}{\sigma^2} + \cancel{1}\right) - \cancel{\frac{N}{4\pi}} = 0$$

$$\log\frac{M^2}{\sigma^2} = \frac{1}{2g^2}\cdot\frac{4\pi}{N}$$

$$\sigma^2 = M^2 \exp\left[-\frac{2\pi}{g^2 N}\right]$$

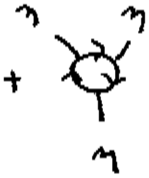
so the potential is minimized at

$$\sigma = \pm M e^{-\pi/g^2 N}$$

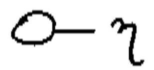
For $g^2 N \ll 1$, $|\sigma| \ll M$

A chg in the renormalization condition chgs M to some other constant, but it does not chge the dependence on $g^2 N$.

8.) Now let $\sigma \to \underbrace{\sigma}_{\text{constant}} + \underbrace{\eta(x)}_{\text{variable with } \eta(k=0)=0}$

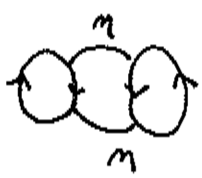
$$\log\det(i\partial\!\!\!/ - \sigma - \eta) = \log\det(i\partial\!\!\!/ - \sigma) +$$  $+$  $+$ 

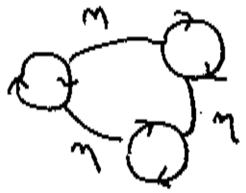
with the propagator $\longrightarrow = \dfrac{i}{k\!\!\!/ - \sigma}$

since η has no $k=0$ mode  $= 0$

then the perturbative corrections to the effective potential come from integrating over η. The propagator for η is

$$\mathcal{L}_0 = i\int d^2x \, \frac{-1}{2g^2}\eta^2 \quad \to \quad \overbrace{\eta\,\eta} = g^2$$

then  $\sim N^2 \cdot (g^2)^2 = g^2 \cdot N \cdot (g^2 N)$



$$= (g^2)^3 N^3 = (g^2 N)^3$$

$$= (g^2)^3 N^2 = g^2 (g^2 N)^2$$

The result of part (e) gave $V_{eff} \sim N^2 f(g^2 N)$

These corrections give g^2 $(g^2)^2$ etc. times this.

So for $N \to \infty$, $g^2 N$ fixed, the result of part (e) becomes <u>exact</u>, even though it is nonperturbative.