Physics 332 – Problem Set # 1

Solutions

1.) a.) $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\not{\partial} - M)\psi$

$$- ig\bar{\psi}\gamma^5\psi\phi$$

the theory has a parity symmetry under which $\phi \to -\phi$

g is a dimensionless coupling constant

– The degree of divergence of a diagram with N_ψ external ψ lines, N_ϕ external ϕ lines, P_ψ P_ϕ propagators, V vertices, L loops is

$$D = 4L - P_\psi - 2P_\phi$$

$$L = P_\psi + P_\phi - V + 1$$

so

$$D = 3P_\psi + 2P_\phi - 4V + 4$$

$$V = 2P_\phi + N_\phi \qquad 2V = 2P_\psi + N_\psi$$

$$3V = 3P_\psi + \tfrac{3}{2}N_\psi$$

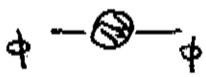
so finally

$$D = 4 - N_\phi - \tfrac{3}{2}N_\psi$$

The superficially divergent amplitudes are:

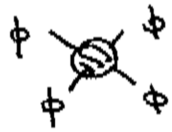

$D = 4$ vacuum energy → ignore

$D = 3$ $= 0$ by parity



$D = 2$ $= A p^2 + B$ 2 divergent terms

$D = 1$ $= 0$ by parity



$D = 0$ $= C$ 1 divergent term

$D = 1$ $= D\not{p} + E$ 2 divergent terms.

$D = 0$ $= F\gamma^5$ 1 divergent term

We need a counterterm for each divergent amplitude:

$$\delta\mathcal{L} = \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{1}{4!}\delta_\lambda\phi^4$$

$$+ \delta_\psi\bar{\psi}i\not{\partial}\psi - \delta_M\bar{\psi}\psi - i\delta_g\bar{\psi}\gamma^5\psi\phi$$

Notice that, even though we omitted the ϕ^4 term in $\mathcal{L}$, we need the counterterm in $\delta\mathcal{L}$. The ϕ^4 interaction is generated by quantum corrections.

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b.) Now compute the diagrams that diverge in 1 loop and determine the counterterms:

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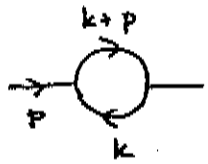
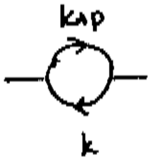


$+$

$(i p^2 \delta_\phi - i\delta_m)$



Fermion loop

$$= g^2 (-1) \int \frac{d^d k}{(2\pi)^d} \operatorname{tr}\left[\gamma^5 \frac{i(\not k + \not p + M)}{(k+p)^2 - M^2} \gamma^5 \frac{i(\not k + M)}{k^2 - M^2} \right]$$

$$= g^2 \int \frac{d^d k}{(2\pi)^d} \frac{\operatorname{tr} \gamma^5 (\not k + \not p + M) \gamma^5 (\not k + M)}{[(k+p)^2 - M^2][k^2 - M^2]}$$

$$\frac{1}{((k+p)^2 - M^2)(k^2 - M^2)} = \int_0^1 dx \frac{1}{[\mathbb{k}^2 + x(1-x)p^2 - M^2]^2}$$

where $\mathbb{k} = k + xp \qquad k = \mathbb{k} - xp$

$k + p = \mathbb{k} + (1-x)p$

$$\operatorname{tr} \gamma^5 (\not k + \not p + M) \gamma^5 (\not k + M) = \operatorname{tr}\left[(\not{\mathbb{k}} + (1-x)\not p) + M\right]\left[-(\not{\mathbb{k}} - x\not p) + M\right]$$

$$= -4\, [\mathbb{k} + (1-x)p]\cdot[\mathbb{k} - xp] + 4M^2$$

$$= -4\mathbb{k}^2 + 4x(1-x)p^2 + 4M^2$$

so

$$= 4g^2 \int_0^1 dx \int \frac{d^d \mathbb{k}}{(2\pi)^d} \frac{[-\mathbb{k}^2 + x(1-x)p^2 + M^2]}{[\mathbb{k}^2 + x(1-x)p^2 - M^2]^2}$$

Now
$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2-\Delta]^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \sim \frac{i}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{-k^2}{[k^2-\Delta]^2} = \frac{i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} \cdot \frac{d}{2}$$

$$= \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{1}{1-d/2} \Delta^{1} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\sim \frac{i}{(4\pi)^2} (-2\Delta) \frac{2}{\epsilon} \qquad \text{near } d=4$$

so the divergent part of this graph is:

$$= 4g^2 \int_0^1 dx \; \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \left\{ -2[M^2 - x(1-x)p^2] + [x(1-x)p^2 + M^2] \right\}$$

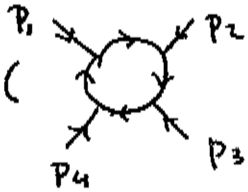
$$\int_0^1 dx \; x(1-x) = \frac{1}{6}$$

$$= \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \, 4g^2 \left(-M^2 + \frac{1}{2} p^2\right)$$

then, to cancel this:

$$\delta_\phi = -\frac{4g^2}{(4\pi)^2} \frac{1}{\epsilon} \qquad \delta_m = -\frac{8g^2}{(4\pi)^2} M^2 \frac{1}{\epsilon}$$

If we had included a $\frac{\lambda \phi^4}{4!}$ term in $\mathcal{L}$, as we should have, we would have had also the diagram , so that
$$\delta_m = \left(-\frac{8g^2 M^2}{(4\pi)^2} - \frac{\lambda m^2}{(4\pi)^2}\right) \frac{1}{\epsilon}$$

 $=$ (

+ other orderings) + 

$-i\delta_2$

there are $3! = 6$ diagrams in which the external lines appear in different orders.

To evaluate the divergent part, we can set all external momenta equal to zero

$$= \quad 6 \cdot g^4 (-1) \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\left[\frac{i(\not{k}+M)}{(k^2-M^2)} \gamma 5 \right]^4$$

only the $\not{k}$ term gives a divergence

$$= \quad 6 \cdot g^4 (-1) \cdot \int \frac{d^d k}{(2\pi)^d} \, (i^4) \, \mathrm{tr} \, \frac{\not{k}\gamma 5 \not{k}\gamma 5 \not{k}\gamma 5 \not{k}\gamma 5}{[k^2-M^2]^4}$$

$$= \quad -6 g^4 \int \frac{d^d k}{(2\pi)^4} \, \frac{\mathrm{tr} \, \not{k}(-\not{k}) \not{k}(-\not{k})}{[k^2-M^2]^4}$$

$$= \quad -6 g^4 \int \frac{d^d k}{(2\pi)^4} \, \frac{4(k^2)^2}{[k^2-M^2]^4}$$

this has the same divergence as

$$= \quad -24 g^4 \int \frac{d^d k}{(2\pi)^4} \, \frac{1}{(k^2)^2}$$

$$= \quad -24 g^4 \, \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \qquad \text{so} \qquad \delta_2 = -\frac{48}{(4\pi)^2} g^4 \frac{1}{\epsilon}$$

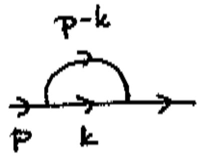
again, if the $\lambda\phi^4$ vertex had been included, we would have found additional diagrams 

so that

$$\delta_\lambda = \left[-\frac{48g^4}{(4\pi)^2} + \frac{3\lambda^2}{(4\pi)^2}\right]\frac{1}{\epsilon}$$

p  $=$  $+$ 

$i(\delta_\psi \not p - \delta_M)$



$$= g^2\int\frac{d^dk}{(2\pi)^d}\,\gamma^5\,\frac{i(\not k+M)}{k^2-M^2}\,\gamma^5\,\frac{i}{(p-k)^2-m^2}$$

$$= -g^2\int\frac{d^dk}{(2\pi)^d}\frac{(-\not k+M)}{(k^2-M^2)((p-k)^2-m^2)}$$

combine w. a Feynman parameter $\quad \mathbb{k} = k - xp$

$k = \mathbb{k} + xp$

$k-p = \mathbb{k} - (1-x)p$

$$= -g^2\int_0^1 dx\int\frac{d^d\mathbb{k}}{(2\pi)^d}\frac{-\not{\mathbb{k}} + x\not p + M}{[\mathbb{k}^2 + x(1-x)p^2 - M^2]^2}$$

$$= \frac{-ig^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{[M^2 - x(1-x)p^2]^{2-\frac{d}{2}}}\,(x\not p + M)$$

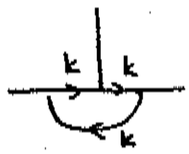
$$= \frac{-ig^2}{(4\pi)^2}\left(\tfrac{1}{2}\not p + M\right)\frac{2}{\epsilon} + \cdots$$

so $\quad \delta_\psi = \dfrac{g^2}{(4\pi)^2}\,\dfrac{1}{\epsilon} \qquad \delta_M = -\dfrac{2g^2}{(4\pi)^2}\, M\, \dfrac{1}{\epsilon}$

Finally

 $=$  $+$ 

$\delta_g\ \gamma^5$



set ext. mom = 0

$$= g^3 \int \frac{d^dk}{(2\pi)^d}\ \gamma^5\ \frac{i\not{k}}{k^2}\ \gamma^5\ \frac{i\not{k}}{k^2}\ \gamma^5\ \frac{i}{k^2}$$

$$= -i\, g^3 \int \frac{d^dk}{(2\pi)^d}\ \frac{(-k^2)\ \gamma^5}{(k^2)^3}$$

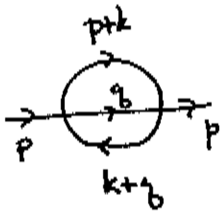
$$= i\, g^3 \int \frac{d^dk}{(2\pi)^d}\ \frac{1}{(k^2)^2}\ \gamma^5$$

$$= i\, g^3\ \frac{i}{(4\pi)^{d/2}}\ \Gamma(2-d/2)\ \gamma^5$$

$$= -\frac{g^3}{(4\pi)^2}\ \frac{2}{\epsilon}\ \gamma^5$$

so $\quad \delta_g = \dfrac{2g^3}{(4\pi)^2}\ \dfrac{1}{\epsilon}$

2.)



$$= \frac{(-i\lambda)^2}{3!}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{i}{(q+k)^2-m^2}\,\frac{i}{q^2-m^2}\,\frac{i}{(p+k)^2-m^2}$$

$$= \frac{-i\lambda^2}{6}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{1}{m^2-(q+k)^2}\,\frac{1}{m^2-q^2}\,\frac{1}{m^2-(p+k)^2}$$

It is easiest to rotate to Euclidean space. At the end of the calculation, we will return to Minkowski space by setting $p^2 \to -p^2$

$$= \frac{-i\lambda^2\,(i)^2}{6}\int \frac{d^dk\, d^dq}{(2\pi)^{2d}} \frac{1}{m^2+(q+k)^2}\,\frac{1}{m^2+q^2}\,\frac{1}{m^2+(p+k)^2}$$

$$= \frac{i\lambda^2}{6}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx\, \frac{1}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\,\frac{1}{[m^2+x(1-x)k^2]^{2-d/2}}\,\frac{1}{m^2+(p+k)^2}$$

$$= \frac{i\lambda^2}{6}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx\,[x(1-x)]^{d/2-2}\int_0^1 dy\,\frac{\cancel{\Gamma(2-d/2)}}{(4\pi)^{d/2}}$$

$$\cdot\; y^{d/2-1}\,\frac{\Gamma(3-d/2)}{\cancel{\Gamma(2-d/2)}\,\underbrace{\Gamma(1)}_{1}}\,\frac{1}{\left[k^2+y(1-y)p^2+\left[y\frac{m^2}{x(1-x)}+(1-y)m^2\right]\right]^{3-d/2}}$$

$$= \frac{i\lambda^2}{6}\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2}\int_0^1 dy\; y^{d/2-1}\,\frac{\cancel{\Gamma(4-d/2)}}{(4\pi)^{d/2}}\cdot\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(3-d)}{\cancel{\Gamma(4-d/2)}}$$

$$\cdot\;\frac{1}{\left[y(1-y)p^2+y\frac{m^2}{x(1-x)}+(1-y)m^2\right]^{3-d}}$$

At this point, we could in fact send $m^2 \to 0$

$$= \frac{i\lambda^2}{6}\int_0^1 dx\,[x(1-x)]^{-\epsilon/2}\int_0^1 dy\; y^{\epsilon/2-1}\,[y(1-y)p^2]^{1-\epsilon}\,\frac{\Gamma(3-d)}{(4\pi)^d}$$

$$= \frac{i\lambda^2}{6}\int_0^1 dx\,[x(1-x)]^{-\epsilon/2}\int_0^1 dy\; y^{-\epsilon/2}\,(1-y)^{1-\epsilon}\,[p^2]^{1-\epsilon}\,\frac{\Gamma(3-d)}{(4\pi)^d}$$

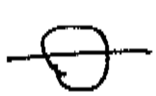
$$\Gamma(3-d) = \frac{1}{3-d}\,\Gamma(4-d) \sim -\frac{1}{\epsilon} \quad \text{as} \quad \epsilon \to 0$$

$$\int_0^1 dy\; y^{-\epsilon/2}(1-y)^{1-\epsilon} \approx \int_0^1 dy\,(1-y) = \frac{1}{2}$$

$$= -i\,\frac{\lambda^2}{12}\,[p^2]^{1-\epsilon}\,\frac{1}{(4\pi)^4}\,\frac{1}{\epsilon} + \cdots$$

$$\simeq -i\,\frac{\lambda^2}{12}\,p^2\,\frac{1}{(4\pi)^4}\left(\frac{1}{\epsilon} - \log p^2 + \cdots\right)$$

return to Minkowski space:

 $= -i\,\frac{\lambda^2}{12}\,\frac{p^2}{(4\pi)^4}\cdot\left(-\frac{1}{\epsilon} + \log(-p^2) + \cdots\right)$

is manifestly independent of p thus

$= -i\,(\text{const})\,m^2 \longrightarrow 0$ as $m^2 \to 0$

3.) We would like to evaluate  in ϕ^4 theory for $s \to \infty$, t fixed. This implies

$$u = 4m^2 - s - t \sim -s \quad \text{as } s \to \infty$$

 $= -i\lambda$

•  $= i\dfrac{\lambda^2}{2(4\pi)^2}\displaystyle\int_0^1 dx \; \log\left[\frac{M^2}{m^2 - x(1-x)s}\right]$ + divergent constant

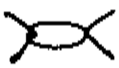
where m is the boson mass, M is the $\overline{MS}$ regulator mass. The divergence is cancelled by a part of . The leading term for large s is

$$= -i\frac{\lambda^2}{2(4\pi)^2}\log s$$

 $\sim \log t/M^2$ + divergent which is $O(1)$ in $\log s$

 $\sim -i\dfrac{\lambda}{2(4\pi)^2}\log u \sim -i\dfrac{\lambda}{2(4\pi)^2}\log s$

so

 +  +  + 

$$= -i\frac{\lambda}{(4\pi)^2}\log s + \dots$$

Now evaluate the 2-loop diagrams. I will start with the s-channel diagrams:

Additional contributions come from the t- and u-channel diagrams.

$s=P^2$ ↑ 

$$= (-i\lambda)^3 \left[\frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(k+P)^2-m^2} \right]^2$$

$$= i\lambda^3 \left[\frac{1}{2} \frac{i}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \frac{1}{[m^2-x(1-x)P^2]^{2-d/2}} \right]^2$$

$$= i\lambda^3 \left[\frac{1}{2} \frac{i}{(4\pi)^2} \left\{ \frac{2}{\epsilon} + \dots \right\} \cdot \left\{ 1 - \frac{\epsilon}{2} \lg s + \frac{1}{2}\left(\frac{\epsilon}{2}\right)^2 \lg^2 s + \dots \right\} \right]^2$$

The second line comes from $\frac{1}{s^{\epsilon/2}} = \exp\left[-\frac{\epsilon}{2} \lg s\right]$

$$= i\lambda^3 \frac{1}{4} \frac{(-1)}{(4\pi)^4} \left(\frac{2}{\epsilon}\right)^2 \left\{ 1 - \epsilon \lg s + \frac{\epsilon^2}{2} \lg^2 s + \dots \right\}$$

The finite term with the strongest s dependence is

$$\sim \quad -i \frac{\lambda^3}{4(4\pi)^2} \cdot 2 \cdot \lg^2 s$$

Something is odd here. If we had made  finite by adding the counterterm before squaring, we would have found:

$$(-i\lambda)^3 \left[\frac{1}{2} \frac{i}{(4\pi)^2} \log \frac{M^2}{s} + \cdots \right]^2$$

$$\sim \quad \frac{-i\lambda^3}{4(4\pi)^2} \log^2 s$$

The difference is accounted for by the contribution from the counterterm diagrams. Recall that

 $= \; -i\,\delta_\lambda \; = \; -i \dfrac{3\lambda^2}{2(4\pi)^2} \cdot \dfrac{2}{\epsilon}$

of this, $\frac{1}{3}$ cancels the divergence of the s-channel subdiagram.

Then

 $+$ 

$$= \left(-i \frac{\lambda^2}{2(4\pi)^2} \frac{2}{\epsilon}\right)(-i\lambda) \cdot \overset{\text{2 diagrams}}{\overset{\downarrow}{2}} \cdot \underset{\text{sym factor of loop}}{\underset{\uparrow}{\frac{1}{2}}} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(k+p)^2-m^2}$$

$$\cong \frac{\lambda^3}{(4\pi)^2} \frac{1}{\epsilon} \frac{i}{(4\pi)^2} \Gamma(\epsilon/2) \cdot \frac{1}{s^{\epsilon/2}}$$

$$\cong \frac{+i\lambda^3}{(4\pi)^4} \frac{2}{\epsilon^2} \left[1 - \frac{\epsilon}{2} \log s + \frac{\epsilon^2}{8} \log^2 s + \cdots \right]$$

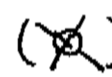
So

 $+$  $+$ 

$$= -i\frac{\lambda^3}{(4\pi)^4}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\lg s + \frac{1}{2}\lg^2 s + \cdots\right]$$

$$-i\frac{\lambda^3}{(4\pi)^4}\left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\lg s - \frac{1}{4}\lg^2 s + \cdots\right]$$

$$= -i\frac{\lambda^3}{(4\pi)^4}\left[\text{(divergent constant)} + \frac{1}{4}\lg^2 s\right]$$

cancelled by ()$_{O(\lambda^3)}$

Note that the $\frac{1}{\epsilon}\lg s$ terms cancel automatically, as they must.

The full contribution from the counterterm diagrams is

 $+$  $= -i\frac{3\lambda^3}{(4\pi)^2}\left[(\text{const}) + \frac{[illegible]}{\epsilon}\lg s - \frac{1}{4}\lg^2 s\right]$

Now we must evaluate

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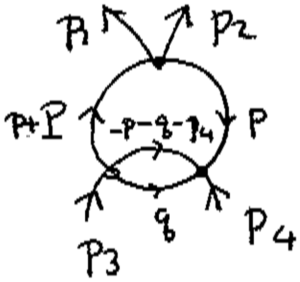


$P^2 = (p_1+p_2)^2 = s$

$$= (-i\lambda)^3 \cdot \frac{1}{2} \cdot \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{i}{(p+P)^2-m^2} \; \frac{i}{p^2-m^2} \; \frac{i}{q^2-m^2} \; \frac{i}{(p+q+p_4)^2-m^2}$$

$$= \frac{i\lambda^3}{2}\int \frac{d^dp\,d^dq}{(2\pi)^{2d}} \int_0^1 dz \frac{1}{[p^2+2zp\cdot P+zP^2-m^2]^2}\int_0^1 dx \frac{1}{[q^2+x(1-x)(p+p_4)^2-m^2]^2}$$

$$= \frac{i\lambda^3}{2}\int\frac{d^dp}{(2\pi)^d}\int_0^1 dz\,dx \frac{1}{[p^2+2zp\cdot P+zP^2-m^2]^2}\;\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{[m^2-x(1-x)(p+p_4)^2]^{2-d/2}}$$

It will be useful to rotate to Euclidean space, rotate back at the end

$$= \frac{i\lambda^3}{2}\cdot\frac{i}{(4\pi)^{d/2}}\Gamma(2-d/2)\cdot i\int_0^1 dz\int_0^1 dx\;[x(1-x)]^{-\epsilon/2}$$

$$\cdot\int\frac{d^dp_{(E)}}{(2\pi)^d}\;\underbrace{\frac{1}{[p^2+2zp\cdot P+zP^2+m^2]^2}}_{(1-y)}\;\underbrace{\frac{1}{\left[\frac{m^2+x(1-x)(p+p_4)^2}{x(1-x)}\right]^{2-d/2}}}_{y}$$

assign a Feynman parameter

the denominator is:

$$\text{Denom} = y\left((p+p_4)^2 + \frac{m^2}{x(1-x)} \right) + (1-y)\left(p^2 + 2z\,p\cdot P + z P^2 + m^2 \right)$$

$$= p^2 + 2p\cdot\left(y p_4 + (1-y) z P \right) + y p_4^2 + (1-y) z P^2$$
$$+ \left[y \frac{1}{x(1-x)} + (1-y) \right] m^2$$

$$= \mathbb{P}^2 - \left(y p_4 + (1-y) z P \right)^2 + y p_4^2 + (1-y) z P^2 + m^2\left[\frac{y}{x(1-x)} + (1-y) \right]$$

$$= \mathbb{P}^2 + y(1-y) p_4^2 + (1-y) z P^2 - (1-y)^2 z^2 P^2 - 2y(1-y) z\, p_4\cdot P$$
$$+ m^2\left[\frac{y}{x(1-x)} + (1-y) \right]$$

$$= \mathbb{P}^2 + y(1-y) z (P-p_4)^2 + y(1-y)(1-z) p_4^2$$
$$+ P^2\left[(1-y) z - y(1-y) z - (1-y)^2 z^2 \right]$$
$$+ m^2\left[\frac{y}{x(1-x)} + (1-y) \right]$$

$$= \mathbb{P}^2 + y(1-y) z\, p_3^2 + y(1-y)(1-z)\, p_4^2 + m^2\left[\frac{y}{x(1-x)} + (1-y) \right]$$
$$+ P^2\left[(1-y)^2 z (1-z) \right]$$

$$\equiv \mathbb{P}^2 + \Delta$$

then  $= -\dfrac{i\lambda^3}{2} \dfrac{1}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) \displaystyle\int_0^1 dx\, dz\, \left[x(1-x) \right]^{-\varepsilon/2}$

$$\int_0^1 dy\, (1-y)\, y^{\varepsilon/2 - 1} \frac{\Gamma(4-\frac{d}{2})}{\Gamma(2-\frac{d}{2})\,\Gamma(2)} \int \frac{d^d\mathbb{P}}{(2\pi)^d} \frac{1}{\left[\mathbb{P}^2 + \Delta \right]^{4-d/2}}$$

$$= -i\,\frac{\lambda^3}{2}\,\frac{1}{(4\pi)^{d_2}}\,\cancel{\Gamma(2-d_2)}\int_0^1 dx\,dz\,[x(1-x)]^{-\varepsilon/2}\int dy\,(1-y)\,y^{\varepsilon/2-1}$$

$$\cdot\ \frac{\Gamma(4-d_2)}{\cancel{\Gamma(2-d_2)}}\ \frac{1}{(4\pi)^{d_2}}\ \frac{\Gamma(4-d)}{\cancel{\Gamma(4-d_2)}}\ \frac{1}{[\Delta]^{4-d}}$$

$$= -i\,\frac{\lambda^3}{2}\,\frac{1}{(4\pi)^d}\,\Gamma(4-d)\int_0^1 dx\,dz\,[x(1-x)]^{-\varepsilon/2}\int_0^1 dy\,(1-y)\,y^{\varepsilon/2-1}\,\frac{1}{[\Delta]^{4-d}}$$

now $p_3^2 = p_4^2 = m^2$ for an on-shell scatt amplitude

for $s\to\infty$ we can ignore m^2. then

$$\Delta \cong P^2\,(1-y)^2 z(1-z) = -s\,(1-y)^2 z(1-z)$$

from Euclidean space.

we have

$$\frac{1}{[-s]^{\varepsilon}} = 1 - \varepsilon\,\lg(-s) + \frac{\varepsilon^2}{2}\lg^2(-s) + \dots$$

to get a $\lg^2(-s)$, we need to expand to $\mathcal{O}(\varepsilon^2)$, so we need a coefficient $\frac{1}{\varepsilon^2}$. this comes from

$$\Gamma(4-d) \cong \frac{1}{\epsilon} \qquad \int_0^1 dy\,y^{\varepsilon/2-1} = \frac{2}{\epsilon}$$

so

$$= -i\,\frac{\lambda^3}{2}\,\frac{1}{(4\pi)^4}\,\frac{1}{\epsilon}\cdot 1\cdot\frac{2}{\epsilon}\cdot\left[1-\epsilon\lg(-s)+\frac{\varepsilon^2}{2}\lg^2 s\right]$$

$$= -i\,\frac{\lambda^3}{2}\,\frac{1}{(4\pi)^4}\left[\lg^2 s - \frac{2}{\epsilon}\lg s + \dots\right]$$

so

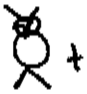
 $$= -i\,\frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon}\lg s + \frac{1}{2}\lg^2 s\right]$$

similarly

 $$= -i\,\frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon}\lg s + \frac{1}{2}\lg^2 s\right]$$

the contributions computed earlier were:

 $$= -i\,\frac{\lambda^3}{(4\pi)^4}\left[-\frac{1}{\epsilon}\lg s + \frac{1}{2}\lg^2 s\right]$$

 $+$  $$= -i\,\frac{\lambda^3}{(4\pi)^4}\left[+\frac{3}{\epsilon}\lg s - \frac{3}{4}\lg^2 s\right]$$

in all, the s-channel diagrams give

$$-i\,\frac{\lambda^3}{(4\pi)^4}\cdot\frac{3}{4}\lg^2 s$$

the t-channel diagrams give $\quad -\frac{i\lambda^3}{(4\pi)^4}\cdot\frac{3}{4}\lg^2 t \sim O(1)$

 $+ \cdots$

the u-channel diagrams give $\quad -i\,\frac{\lambda^3}{(4\pi)^4}\frac{3}{4}\lg^2 u \sim -i\,\frac{\lambda^3}{(4\pi)^4}\frac{3}{4}\lg^2 s$

 $+ \cdots$

in all

$$\left(\quad \right)_{2\text{-loop}} \sim -i \frac{\lambda^3}{(4\pi)^4} \frac{3}{2} \lg^2 s$$

$$\sim -i\lambda - i \frac{\lambda^2}{(4\pi)^2} \lg s - i \frac{3}{2} \frac{\lambda^3}{(4\pi)^4} \lg^2 s$$