Physics 332 – Problem Set # 5

(due Wednesday, May 17)

1. Consider scalar electrodynamics:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2 \tag{1}$$

where ϕ is a complex-valued field and $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$.

(a) Remind yourself that, classically, the A_μ field acquires a mass when ϕ acquires a classical vacuum expectation value.

(b) Write the functional integral for A_μ and ϕ in the R_ξ gauge for a fixed background $\varphi_{cl} = \langle\phi\rangle$. Notice that the dependence on φ_{cl} simplifies for $\xi = 0$ (Lorenz gauge).

(c) Compute the 1-loop quantum correction to the effective potential for φ_{cl}. Renormalize by minimal subtraction, introducing a renormalization scale M. For minimal credit, perform this calculation at $\xi = 0$. For extra credit, show that, to first order in perturbation theory, the answer is independent of ξ. Anyway, from here on, it is simplest to set $\xi = 0$.

(d) Take the limit in which the renormalized mass m^2 goes to zero. Consider this theory for small values of λ, such that, parametrically, $\lambda \sim e^4$. Show that there are parameters in this region such that the $m^2 = 0$ theory has spontaneous symmetry breaking; thus, there is symmetry breaking due to quantum corrections.

(e) Compute the renormalization group β functions for this theory. Since the β functions are gauge-invariant, you could do this in Feynman gauge ($\xi = 1$), but it is somewhat easier to use $\xi = 0$. Either way, you should find:

$$\beta_e = \frac{e^3}{48\pi^2} \qquad \beta_\lambda = \frac{1}{24\pi^2}(5\lambda^2 - 18e^2\lambda + 54e^4) \tag{2}$$

Sketch the renormalization group flows (toward the infrared) in the (λ, e^2) plane. Show that every renormalization group trajectory actually passes through $\lambda = 0$ and goes into the region $\lambda < 0$.

(f) Consider the approximation to the $m^2 = 0$ effective potential given by

$$V_{eff} = \frac{1}{6}\overline{\lambda}(\varphi_{cl})(Z(\varphi_{cl})^2\varphi_{cl}^4 \tag{3}$$

Show that this expression has spontaneous symmetry breaking, just as we found in part(d).

(g) Now consider $m^2 \neq 0$. Show that the theory has a first-order phase transition as a function of m^2.

2. Apply the methods of this problem to the Glashow-Salam-Weinberg model of weak interactions.

   (a) Compute the effective potential for the Higgs field to 1-loop order, ignoring all effects of quark masses but including the contributions of gauge fields.

   (b) Show that the theory has a first-order phase transition as a function of the renormalized Higgs mass parameter μ^2.

   (c) Show that this result implies a lower bound on the physical mass of the Higgs boson (the 'Linde-Weinberg bound'). Compute the bound to leading order in coupling constants.

   (d) Now add in the contribution of the top quark. Show that, when the top quark mass is sufficiently heavy, the symmetry-breaking effect found in part (b) goes away. However, another pathology develops, in which, when m_t is sufficiently large, the effective potential becomes negative at very large field values and causes an instability of the model. Estimate the value of the top quark mass, as a function of the Higgs boson and W boson masses, at which this instability takes place.