Physics 332 – Final Exam

This exam is due at 3:30 pm on Friday, June 16. Please hand it in to Nav Sivanandam in Varian 365. If you have any questions about the exam, please contact me by email at mpeskin@slac.stanford.edu. If errata are reported, I will announce them on the course Web page.

Please do not collaborate on this exam. Please return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed.

The exam is worth a total of 100 points. The distribution of points is indicated below.

Many of the calculations on this exam can be found in the literature. But in order to get the answers from the literature, you will need to spend a lot of time translating notation. What fun is that? If you make strong use of a reference other than the class textbook and notes, please cite the reference in your solutions.

Quantum field theory models with supersymmetry (fermion-boson symmetry) have many interesting renormalization properties. In this exam, you will work out several of these.

a. (20 points) Here is a simple model of massless scalars and fermions with supersymmetry: Let $\phi(x)$ be a complex scalar field, and let $\psi(x)$ be a 2-component left-handed fermion. Consider the Lagrangian

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi + \psi^\dagger i\sigma\cdot\partial\psi \\ &\qquad + \frac{1}{2}(2y)(\psi^T\epsilon\psi\phi - \psi^\dagger\epsilon\psi^*\phi^*) - z(\phi^*\phi)^2 \end{aligned} \tag{1}$$

where y and z are real-valued coupling constants. Notice that the fermions are contracted in the second line in a Lorentz-invariant way:

$$\psi^T\epsilon\psi = \epsilon_{\alpha\beta}\psi_\alpha\psi_\beta \ ; \qquad (\psi^T\epsilon\psi)^\dagger = -\psi^\dagger\epsilon\psi^* = \epsilon_{\alpha\beta}\psi^\dagger_\alpha\psi^\dagger_\beta \tag{2}$$

According to Peskin and Schroeder, Problem 3.5, this model is supersymmetric if $z = y^2$.

Compute the β functions of y and z in this model. Show that the relation $z = y^2$ is preserved by the renormalization group.

b. (10 points) Compute the scalar self-energy in this model. Show that it is not quadratically divergent—and that it even vanishes identically—when $z = y^2$.

c. (15 points) Here is a supersymmetric version of Yang-Mills theory: Let ψ_i be left-handed 2-component fermions in the representation r of the gauge group G, and let ϕ_i be complex-valued scalars in the same representation. Let A^a_μ be the Yang-Mills fields, and let λ^a be left-handed 2-component fermions in the adjoint representation of G. Let

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F^a_{\mu\nu})^2 + \lambda^\dagger i\sigma\cdot D\lambda + D_\mu\phi_i^* D^\mu\phi_i + \psi_i^\dagger i\sigma\cdot D\psi_i \\ &\quad +\sqrt{2}h(\phi_i^*\lambda^{aT}t^a\epsilon\psi_i - \psi_i^\dagger\epsilon\lambda^{a*}t^a\phi_i) - \frac{1}{2}k(\phi_i^* t^a\phi_i)^2 \ , \qquad (3)\end{aligned}$$

where t^a are representation matrices of G in the representation r. The label $i = 1,\ldots,n_f$; remember that the fermions are Weyl, not Dirac. In the last term, the sum over i is done before squaring. There are three coupling constants: g (the Yang-Mills coupling), h, and k. The theory is supersymmetric if $h = g$, $k = g^2$.

Compute the scalar self-energy in this model. Use Feynman gauge. Show that the ϕ self-energy is not quadratically divergent when the supersymmetry relations of coupling constants are satisfied. Show that this self-energy does not vanish identically in dimensional regularization, but that it does vanish if we set

$$g_{\mu\nu}g^{\mu\nu} = 4 \qquad (4)$$

as we continue in dimensionality. This is a different conventional regularization prescription called ‘dimensional reduction’. Only the latter prescription for the regulator preserves supersymmetry.

d. (15 points) Compute the β function of g using formulae derived in the course. Compute the β functions of h, k explicitly. Show that the relations $h = g$, $k^2 = g$ are preserved by the renormalization group.

e. (15 points) Next, build a supersymmetric theory with one quark coupled to a Higgs boson. Let ψ_q be a left-handed fermion in the (3,2) of $SU(3)\times SU(2)$, and let ϕ_q be the corresponding complex scalar. Let $\psi_{\overline{q}}$ be a left-handed fermion in the ($\overline{3}$,1) of $SU(3))\times SU(2)$ and let $\phi_{\overline{q}}$ be the corresponding complex scalar, and let ψ_h, ϕ_h be the Higgs fermion and scalar. It is useful to visualize the $SU(3)$ and $SU(2)$ indices by thinking of $\phi_{\overline{q}}$, $\psi_{\overline{q}}$ as 3-dimensional row vectors, ϕ_h, ψ_h as 2-dimensional column vectors, and ϕ_q, ψ_q as 3×2 matrices. In the following, t^a is the color representation matrix in the 3 of $SU(3)$; in the $\overline{3}$, $t^a_{\overline{3}} = -t^{aT}$.

The supersymmetric interaction terms for this model (including QCD but not $SU(2)$ gauge interactions) are

$$\begin{aligned}\mathcal{L}_{int} &= +\sqrt{2}g(\phi_q^*\lambda^{aT}t^a\epsilon\psi_q - \psi_{\overline{q}}^T t^a\epsilon\lambda^a\phi_{\overline{q}}^* - h.c.) - \frac{1}{2}g^2(\phi_q^* t^a\phi_q - \phi_{\overline{q}}t^a\phi_{\overline{q}}^*)^2 \\ &\quad +y(\psi_{\overline{q}}\epsilon\psi_q\phi_h + \psi_{\overline{q}}\epsilon\phi_q\psi_h + \phi_{\overline{q}}\psi_q\epsilon\psi_h + h.c.) \\ &\quad -y^2(|\phi_{\overline{q}}\phi_q|^2 + |\phi_{\overline{q}}\phi_h|^2 + |\phi_q\phi_h|^2) \qquad (5)\end{aligned}$$

The coupling y is the Higgs Yukawa coupling, which would be large, for example, for the top quark.

Compute the β functions due to QCD (terms of order $g^2 y$) for the three different 2-fermion-1-scalar interaction terms in the second line. There is no obvious reason why the terms with one Higgs scalar should have the same renormalization as the terms with one Higgs fermion. But, show that it is so.

f. (15 points) Add to this theory supersymmetry-violating mass terms for the quarks and the Higgs:

$$\Delta\mathcal{L} = -M_q^2|\phi_q|^2 - M_{\overline{q}}^2|\phi_{\overline{q}}|^2 - M_h^2|\phi_h|^2 \qquad (6)$$

where the M_i^2 are positive and are equal at the renormalization scale μ. We have already seen that, in a supersymmetric model, these operators do not receive additive radiative corrections. However, they do have γ functions and so the coefficients undergo a renormalization group evolution. Compute the renormalization group equations for the M_i^2 parameters. Show that, if the 'gauginos' λ^a are massless, the only effect in one-loop order comes from the interactions in the third line of (5). Show that this effect makes the M_i^2 *decrease* as we move toward the infrared.

g. (10 points) Describe the qualitative nature of this theory in the infrared. Is there symmetry breaking? Which field gets a vacuum expectation value?

Self-evaluation: To record a satisfactory performance on this exam, please complete at least through part (c). Prospective supersymmetry theorists should slog through to the end.