Physics 332 – Final Exam
Solutions

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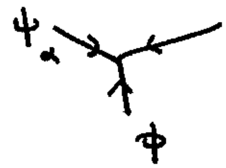
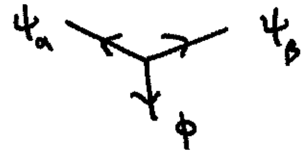
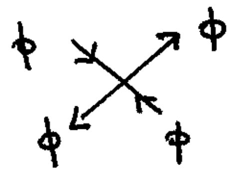
a.) The Feynman rules for this theory are



$= 2iy\,\varepsilon_{\alpha\beta}$



$= -2iy\,\varepsilon_{\alpha\beta}$



$= -4iz$

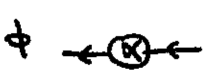
The signs in the fermion terms are tricky. To be safe, I'll work them out by making explicit contractions. The propagators are:

$$\overline{\psi_\alpha(p)\,\psi^\dagger_\beta(-p)} = \frac{i(\sigma\cdot p)_{\alpha\beta}}{p^2} = -\,\overline{\psi^\dagger_\beta(-p)\,\psi_\alpha(p)} = +\,\overline{\psi^\dagger_\beta(p)\,\psi_\alpha}$$

(odd under $p \to -p$)

$$\overline{\phi(p)\,\phi^*(-p)} = \frac{i}{p^2}$$

The counterterms are:



$= ip^2\,\delta_{Z\phi}$

$= i\bar{\sigma}\cdot p\,\delta_{Z\psi}$


$= 2i\,\delta_y\,\varepsilon_{\alpha\beta}$

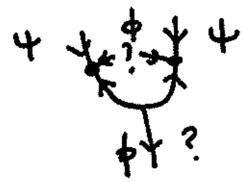

$= -2i\,\delta_y\,\varepsilon_{\alpha\beta}$


$= -4i\,\delta_Z$

First, compute the counterterms δ_y, δ_z :

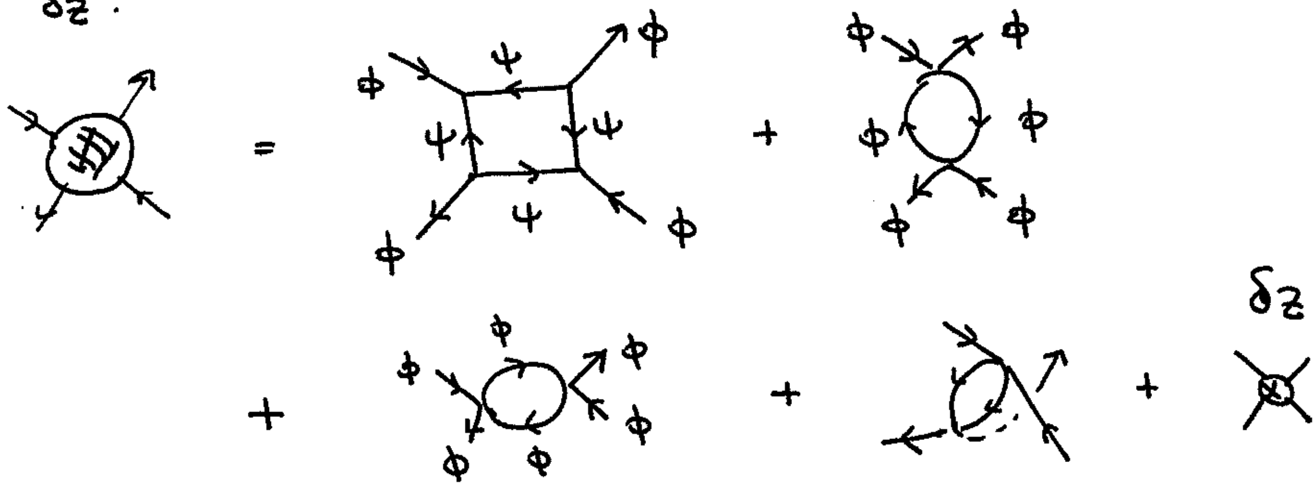
Actually, there are no 1-loop divergences for the vertex

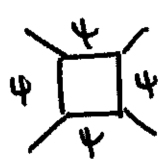
The arrows do not work !



so $\delta_y = 0$

for δ_z :





$$= (2iy)(-2iy)(2iy)(-2iy)$$

$$\cdot\, (\psi^T \epsilon\, \psi\, \phi)(\psi^\dagger \epsilon\, \psi^* \phi^*)(\psi^T \epsilon\, \psi\, \phi)(\psi^\dagger \epsilon\, \psi^* \phi^*)$$

$$\psi(p)\psi^\dagger(p) = \frac{i\sigma\cdot p}{p^2} \qquad \psi^*(p)\,\psi^T(-p) = (-1)\cdot\left[\frac{i\sigma\cdot(-p)}{p^2}\right]^T$$

$$= 16y^4 \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\left[\epsilon\, \frac{i\sigma p}{p^2}\, \epsilon \left(\frac{i\sigma\cdot p}{p^2}\right)^T \epsilon\, \frac{i\sigma\cdot p}{p^2}\, \epsilon\, \underbrace{(-1)\left(\frac{i\sigma\cdot p}{p^2}\right)^T}\right]$$

the last propagator is in the order $\psi^T(-p)\,\psi^*(p)$

now $\quad \epsilon\,(\sigma\cdot p)^T \epsilon = \epsilon^2\, \bar{\sigma}\cdot p = (-1)\cdot\bar{\sigma}\cdot p$

$$\sigma\cdot p\;\bar{\sigma}\cdot p = p^2$$

so

$$= 16y^4 \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\left[(-1)\,\frac{p^2}{p^2 p^2}\cdot(-1)(-1)\,\frac{p^2}{p^2\cdot p^2}\right]$$

$$= -16\cdot 2y^4 \int\frac{d^dp}{(2\pi)^d}\,\frac{1}{[p^2]^2} = (-32y^4)\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)$$

$$= (-4iz)^2 \int\frac{d^dp}{(2\pi)^d}\left(\frac{i}{p^2}\right)^2$$

$$= 16z^2 \int\frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^2} = 16z^2\cdot\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)$$

$$= 16z^2\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)$$

$$= \frac{1}{2}\cdot 16z^2\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)$$

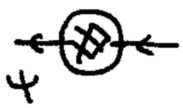
in all:

$$= \left(-32y^4 + \frac{5}{2}\cdot 16z^2\right)\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2) - 4i\delta_z$$

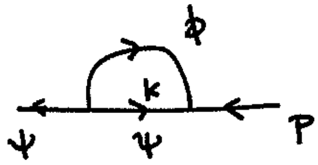
$$= \left(-32y^4 + 40z^2\right)\frac{i}{(4\pi)^2}\log\Lambda^2/M^2 - 4i\delta_z$$

so $\quad \delta_y = 0 \qquad \delta_z = (10z^2 - 8y^4)\,\frac{1}{(4\pi)^2}\log\Lambda^2/M^2$

Now compute the self-energy counterterms:



$=$



$+$

$$\psi^{+}\epsilon\,\overbrace{\psi^{*}\ \psi^{T}}\epsilon\,\psi$$

$$= (-2iy)(2iy)\int\frac{d^dk}{(2\pi)^d}\ \epsilon\,(-1)\left(\frac{i\sigma\cdot k}{k^2}\right)^T\epsilon\ \frac{i}{(p+k)^2} + i\bar{\sigma}\cdot p\,\delta_{Z\psi}$$

$$= 4y^2\int\frac{d^dk}{(2\pi)^d}\ \epsilon(\sigma\cdot k)^T\epsilon\ \frac{1}{k^2(p+k)^2} + i\bar{\sigma}\cdot p\,\delta_{Z\psi}$$

$$= 4y^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\ (-\bar{\sigma}\cdot k)\frac{1}{[k^2+xp\cdot k+xp^2]^2} + \cdots$$

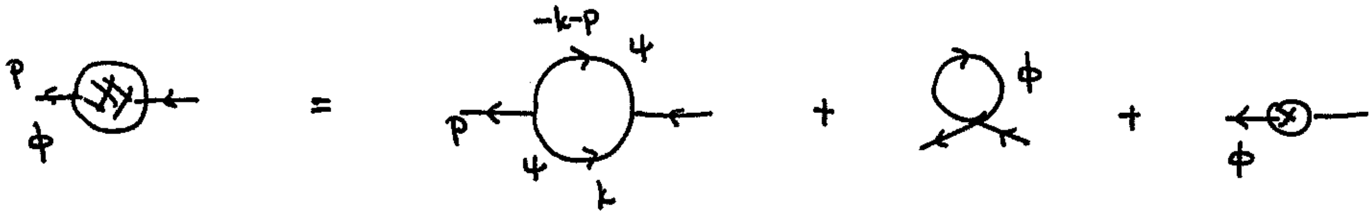
$$\ell = k+xp \qquad k = \ell - xp$$

$$= (-4y^2)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\ \frac{\bar{\sigma}\cdot(\ell-xp)}{[\ell^2-\Delta]^2} + \cdots$$

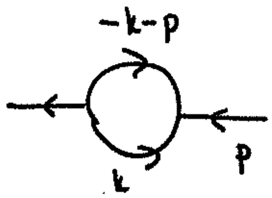
$$= -4y^2\int_0^1 dx\ [-x(\bar{\sigma}\cdot p)]\cdot\frac{i}{(4\pi)^{d/2}}\Gamma(2-d/2)$$

$$= +2y^2\frac{i}{(4\pi)^2}\log\Lambda^2/M^2\cdot\bar{\sigma}\cdot p + i\bar{\sigma}\cdot p\,\delta_{Z\psi}$$

$$\delta_{Z\psi} = -2y^2\frac{1}{(4\pi)^2}\log\Lambda^2/M^2$$



to compute δZ_ϕ, we need the term of order p^2 from the diagrams.
However, for part (b), we also need the term of order $(p^2)^0$.
so, why not compute everything now?



$$\phi^* \psi^\dagger \epsilon \psi^* \quad \psi^T \epsilon \psi \ \phi$$

$$= \frac{1}{2}(-2iy)(2iy)\int \frac{d^dk}{(2\pi)^d} \ \mathrm{tr}\ \epsilon\ (-1)\left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon\ (-1)\ \frac{i\sigma\cdot(-k-p)}{(k+p)^2}$$

$$= 2y^2 \int \frac{d^dk}{(2\pi)^d} \ \mathrm{tr}\ \frac{\epsilon(\sigma\cdot k)^T \epsilon \ \sigma\cdot(k+p)}{k^2\,(k+p)^2}$$

$$= 2y^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \ -2\ k\cdot(k+p)\ \frac{1}{[\ell^2 - x(1-x)p^2]^2}$$

$$\ell = k + xp \qquad k = \ell - xp \qquad k+p = \ell + (1-x)p$$

$$= -4y^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \ \frac{[\ell^2 - x(1-x)p^2]}{[\ell^2 - x(1-x)p^2]^2}$$

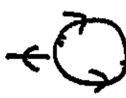
$$= -4y^2 \int_0^1 dx \ \frac{i}{(4\pi)^{d/2}} \left\{ - \frac{\Gamma(1-\frac{d}{2})\ \frac{d}{2}\,[-x(1-x)p^2] - x(1-x)p^2\,\Gamma(2-\frac{d}{2})}{[-x(1-x)p^2]^{2-d/2}} \right.$$

$$= -4y^2 \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{\Gamma(1-\frac{d}{2})\cdot\frac{d}{2} - \Gamma(2-\frac{d}{2})}{[-x(1-x)p^2]^{2-d/2}} \right) [x(1-x)p^2]$$

 $= (-4iz)\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2}$

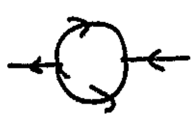
$$= 4z\cdot\frac{-i}{(4\pi)^{d/2}}\,\Gamma(1-d/2)$$

the quadratic divergence of the scalar mass is:

 +  $\sim -4y^2\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{k^4} + 4z\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$

which vanishes when $z=y^2$!

Now continue to $d=4$ to find the term of order p^2.  gives no contribution.

 $= -4y^2\frac{i}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{[-x(1-x)p^2]^{2-d/2}}\,[x(1-x)p^2]$

$$\cdot\left(\frac{d/2}{1-d/2}-1\right)$$

$$= -4y^2\cdot(-3)\cdot\frac{i}{(4\pi)^2}\overbrace{\int_0^1 dx\; x(1-x)}^{1/6}\;p^2\log\Lambda^2/M^2$$

 $= ip^2\,\delta_{Z\phi}$

so $\delta_{Z\phi} = -2y^2\frac{1}{(4\pi)^2}\log\Lambda^2/M^2 = \delta_{Z\psi}$

Now compute the β function for y and z:

$$\beta_y = M\frac{\partial}{\partial M}\left[-\delta_y + \frac{y}{2}\left(\delta_{Z\phi} + 2\delta_{Z\psi}\right)\right]$$

$$= M\frac{\partial}{\partial M}\left[0 + \frac{y}{2}\cdot 3\cdot\left(-2y^2\frac{1}{(4\pi)^2}\log\Lambda^2/M^2\right)\right]$$

$$\beta_y = \frac{6y^3}{(4\pi)^2}$$

$$\beta_z = M\frac{\partial}{\partial M}\left[-\delta_z + \frac{z}{2}\cdot(4\delta_{Z\phi})\right]$$

$$= M\frac{\partial}{\partial M}\left[\frac{8y^4 - 10z^2}{(4\pi)^2}\log\Lambda^2/M^2 + (2z)\left(-2y^2\frac{1}{(4\pi)^2}\log\Lambda^2/M^2\right.\right.$$

$$= (-2)\frac{8(y^4 - z^2)}{(4\pi)^2} + \frac{2\cdot 2z^2}{(4\pi)^2} + \frac{2\cdot 4y^2 z}{(4\pi)^2}$$

$$\beta_z = \frac{1}{(4\pi)^2}\left[16(z^2 - y^4) + 4z^2 + 8y^2 z\right]$$

for $z = y^2$ $\quad\beta_z = \frac{1}{(4\pi)^2}12y^4 = 2y\,\beta_y$

this is just what is required so that

$$\frac{dy}{d\log Q} = \beta_y \qquad \frac{dz}{d\log Q} = \beta_z \quad \text{preserves } z = y^2$$

something seems odd here. If we have supersymmetry, shouldn't we have $\delta_y = 0 = \delta_Z$, $\delta_{Z\phi} = \delta_{Z\psi}$?

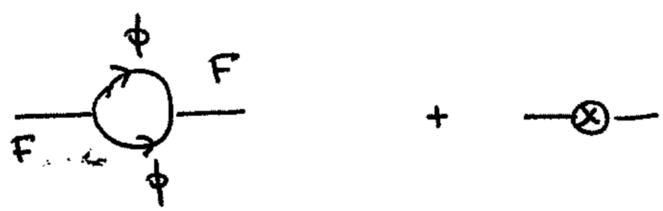
For the explanation, please look at P+S problem 3.5. In that problem, the Lagrangian is written:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger i \bar{\sigma} \cdot \partial \psi + F^\dagger F$$
$$+ y \left(\psi^T \epsilon \psi \phi + h.c. \right) + y F \phi^2 + h.c.$$

The supersymmetry multiplet is actually (ϕ, ψ, F). The propagator for F is

$$\contraction{}{F}{}{F} F F^* = i$$

F has a self-energy :

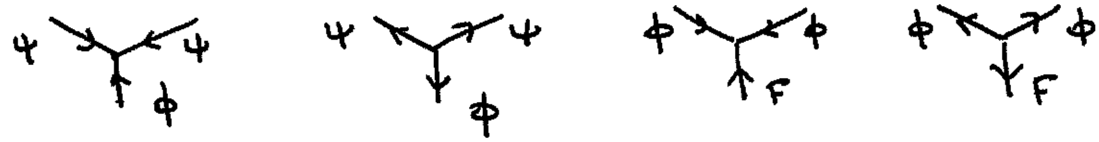


$$= \frac{1}{2} \cdot (2iy)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2} \frac{i}{(k+p)^2} + i\,\delta_{ZF}$$

$$= 2y^2 \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) + i\,\delta_{ZF}$$

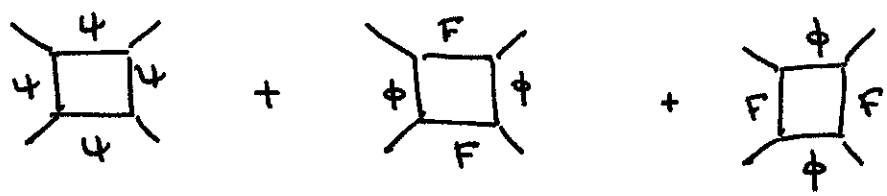
so $$\delta_{ZF} = -2y^2 \frac{1}{(4\pi)^2} \log \Lambda^2/M^2 = \delta_{Z\phi} = \delta_{Z\psi}$$

the vertices of this theory are :



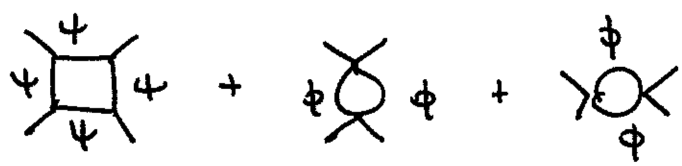
there are no 1-loop corrections to the $\phi\psi\psi$ vertex or to the $F\phi\phi$ vertex. It seems that we can generate a $\phi^2\phi^{*2}$ vertex, but

if $z = y^2$



$= 0$

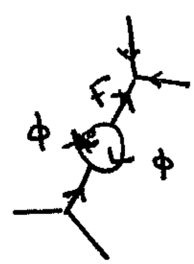
[as we saw on p. 3:



$= 0$]

On p. 3, there is one more graph that does not cancel.

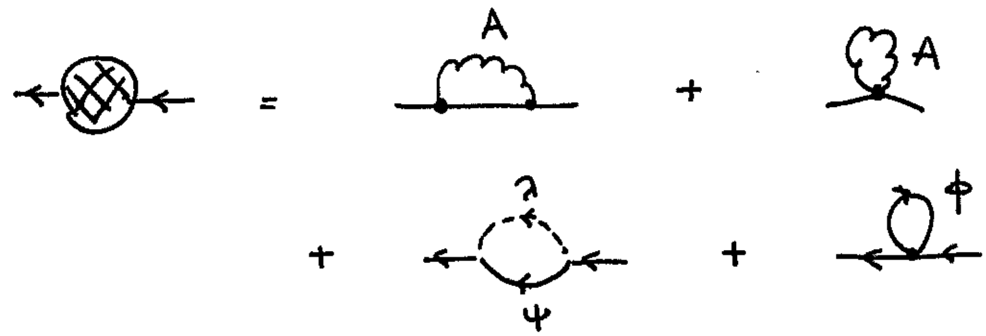
In the theory with F, this is

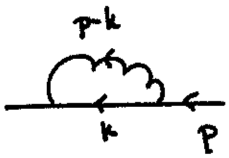


i.e., it is the effect of the F self energy when F is integrated out.

b.) [this was shown on p.6]

c.)





$$= (ig)^2 \int \frac{d^dk}{(2\pi)^d}\, t^a (k+p)_\mu \frac{i}{k^2} (k+p)^\mu t^a \frac{-i}{(p-k)^2}$$

$$= -g^2 C_2(r) \int \frac{d^dk}{(2\pi)^d} \frac{(k+p)^2}{k^2 (k-p)^2}$$

$\ell = k - xp$
$k = \ell + xp$

$$= -g^2 C_2(r) \int dx \int \frac{d^d\ell}{(2\pi)^d} \frac{[\ell + (1+x)p]^2}{[\ell^2 - x(1-x)p^2]^2}$$

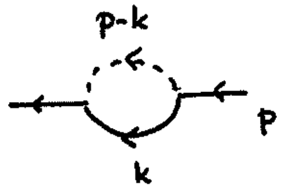
$$= -g^2 C_2(r) \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \left\{ -\frac{\Gamma(1-\frac{d}{2})}{[-x(1-x)p^2]^{1-\frac{d}{2}}} \frac{d}{2} + \frac{\Gamma(2-\frac{d}{2})\,(1+x)^2 p^2}{[-x(1-x)p^2]^{2-\frac{d}{2}}} \right\}$$

$$= +g^2 C_2(r) \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \left\{ \frac{\frac{d}{2}\Gamma(1-\frac{d}{2})}{[-x(1-x)p^2]^{1-\frac{d}{2}}} - \frac{\Gamma(2-\frac{d}{2})\,(1+x)^2 p^2}{[-x(1-x)p^2]^{2-\frac{d}{2}}} \right\}$$

$$= \frac{1}{2}\, 2ig^2 \int \frac{d^dk}{(2\pi)^d}\, g_\mu^{\ \mu} \frac{-i}{k^2}\, t^a t^a$$

$$= (\tfrac{1}{2}) \cdot 2 \cdot 4\, g^2 C_2(r) \left[\frac{-i}{(4\pi)^{d/2}} \Gamma(1-\tfrac{d}{2}) \right]$$

(∵ $g_\mu^{\ \mu} = 4$
≈ DR)



$$\phi^* \lambda^T \epsilon \psi \quad \psi^+ \epsilon \lambda^* \phi$$

$$= (\sqrt{2}\, i h)(-\sqrt{2}\, i h) \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\, \epsilon \left(\frac{i\sigma\cdot k}{k^2}\right) \epsilon \left[\frac{i\sigma(p-k)}{(p-k)^2}\right]^T t^a t^a$$

$$= 2h^2 C_2(r) \int \frac{d^d k}{(2\pi)^d} \frac{2k\cdot(p-k)}{k^2(p-k)^2}$$

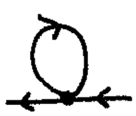
$\mathbb{k} = k - xp$

$k = \mathbb{k} + xp$

$(k-p) = \mathbb{k} - (1-x)p$

$$= 2h^2 C_2(r) \int_0^1 dx \int \frac{d^d \mathbb{k}}{(2\pi)^d} \, 2(\mathbb{k}+xp)\cdot(-\mathbb{k}+(1-x)p) \frac{1}{[\mathbb{k}^2 - x(1-x)p^2]^2}$$

$$= -4h^2 C_2(r) \int_0^1 dx \, \frac{i}{(4\pi)^{d_2}} \left\{ -\frac{\Gamma(1-d_2)\cdot d_2}{[-x(1-x)p^2]^{1-d_2}} - \frac{\Gamma(2-d_2)\, x(1-x)p^2}{[-x(1-x)p^2]^{2-d_2}} \right\}$$



$$= \quad -ik \int \frac{d^d k}{(2\pi)^d} \, t^a \frac{i}{k^2} t^a + t^a \frac{i}{k^2} \underbrace{\mathrm{tr}[t^a]}_{=0}$$

$$= k \, C_2(r) \left[\frac{-i}{(4\pi)^{d_2}} \Gamma(1-d_2)\right]$$

The terms at $p^2=0$ are:

$$\left[\frac{-i}{(4\pi)^{d_2}} \Gamma(1-d_2)\right] \left\{ C_2(r) \right\} \left\{ -g^2 \frac{d}{2} + 4g^2 - 4h^2 \frac{d}{2} + k \right\}$$

$$\underset{\text{at } d=2}{=} \frac{-i}{(4\pi)^{d_2}} \Gamma(1-d_2)\, C_2(r) \left\{ -g^2 + 4g^2 - 4h^2 + k \right\}$$

$$= 0 \quad \text{if} \quad h = g, \; k = g^2$$

d.) For a gauge ~~theory~~ with matter fermions and scalars,

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{\substack{\text{Dirac}\\\text{fermions}}} C(r_i) - \frac{1}{3}\sum_{\substack{\text{complex}\\\text{scalars}}} C(r_j)\right]$$

a Weyl fermion counts $\frac{1}{2}$ of a Dirac fermion. In the theory we have

have	Weyl fermions	λ	$C(r_\lambda) = C(G) = C_2(G)$
		ψ_i	$C(r)$
and	scalars	ϕ_i	$C(r)$

so

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{2}{3}C_2(G) - \frac{2}{3}n_f C(r) - \frac{1}{3}n_f C(r)\right]$$

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right]$$

to compute the β function for h and k, we need the self-energy counterterms for ϕ, ψ, λ

For ϕ, look back at pp. 10-11

$$= g^2 C_2(r)\int_0^1 dx\, \frac{i}{(4\pi)^{d/2}}\left\{\frac{\frac{d}{2}\Gamma(1-\frac{d}{2})}{[-x(1-x)p^2]^{1-\frac{d}{2}}} - \frac{(1+x)^2p^2\,\Gamma(2-\frac{d}{2})}{[\quad]^{2-\frac{d}{2}}}\right\}$$

$$= 4h^2 C_2(r)\int_0^1 dx\, \frac{i}{(4\pi)^{d/2}}\left\{\frac{\frac{d}{2}\Gamma(1-\frac{d}{2})}{[-x(1-x)p^2]^{1-\frac{d}{2}}} + \frac{x(1-x)p^2\,\Gamma(2-\frac{d}{2})}{[\quad]}\right\}$$

  do not contribute to δZ_ϕ.

$$= \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{[-x(1-x)p^2]^{2-d/2}} \Big\{ g^2 C_2(r) \left[-\left(\frac{d/2}{1-d/2}\right) x(1-x)p^2 - (1+x)^2 p^2 \right]$$
$$+ 4h^2 C_2(r) \left[-\left(\frac{d/2}{1-d/2}\right) x(1-x)p^2 + x(1-x)p^2 \right] \Big\}$$

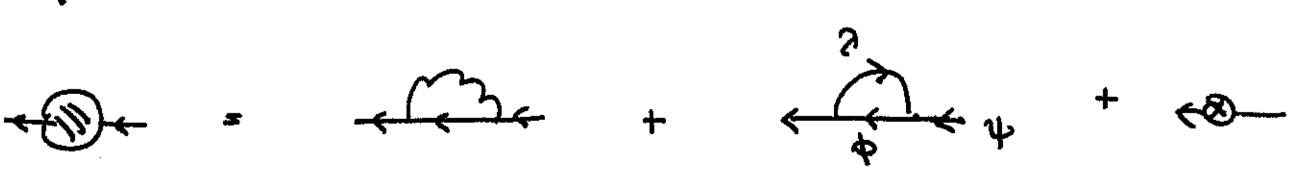
$$= \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \, (\log \Lambda^2/M^2) \, p^2 \Big\{ g^2 C_2(r) \cdot [2x(1-x) - (1+x)^2]$$
$$+ 4h^2 C_2(r) \, [2x(1-x) + x(1-x)] \Big\}$$

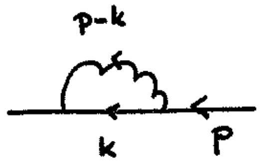
$$= ip^2 \left(\frac{1}{(4\pi)^2} \log \Lambda^2/M^2\right) \left[g^2 C_2(r) \left(\frac{1}{3} - \frac{7}{3}\right) + 4h^2 C_2(r) \cdot \frac{3}{6} \right]$$

$$= ip^2 \, \frac{1}{(4\pi)^2} \log \Lambda^2/M^2 \, \left[-2g^2 C_2(r) + 2h^2 C_2(r) \right]$$

so $$\delta_{Z\phi} = (2g^2 - 2h^2) \, C_2(r) \, \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

for ψ:



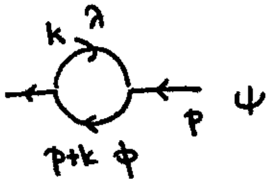


$$= (ig)^2 C_2(r) \int \frac{d^dk}{(2\pi)^d} \, \bar{\sigma}^\mu \frac{i\sigma\cdot k}{k^2} \bar{\sigma}_\mu \frac{-i}{(p-k)^2}$$

$$= -g^2 C_2(r) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{-2\,\bar{\sigma}\cdot k}{[\ell^2 - x(1-x)p^2]^2} \qquad \begin{array}{l} \ell = k - xp \\ k = \ell + xp \end{array}$$

$$= 2g^2 C_2(r) \int_0^1 dx \, \frac{i}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) \frac{\bar{\sigma}\cdot xp}{[-x(1-x)p^2]^{2-d/2}}$$

$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) g^2 C_2(r) \, \bar{\sigma}\cdot p$$



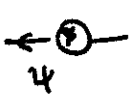
$$\overbrace{\psi^\dagger \epsilon \lambda^* \phi \quad \phi^* \lambda^T}\epsilon \psi$$

$$= (-\sqrt{2}ih)(\sqrt{2}ih) \int \frac{d^dk}{(2\pi)^d} t^a \epsilon (-1) \left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon t^a \frac{i}{(p+k)^2}$$

$$= 2h^2 (-1) \int \frac{d^dk}{(2\pi)^d} C_2(r) \frac{\bar{\sigma}\cdot k}{k^2 (p+k)^2}$$

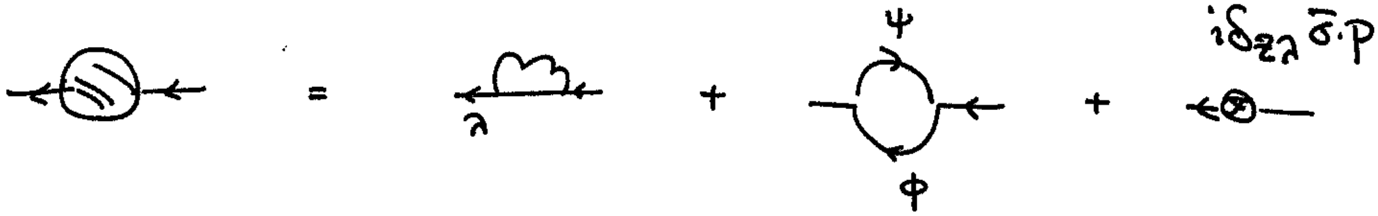
$$= -2h^2 C_2(r) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\bar{\sigma}\cdot \ell - xp}{[\ell^2 - x(1-x)p^2]^2}$$

$$= \frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \cdot [+h^2 C_2(r)] \, \bar{\sigma}\cdot p$$

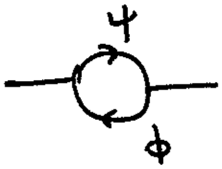


$$= i\bar{\sigma}\cdot p \, \delta Z_\psi \quad \text{so} \quad \delta Z_\psi = -(g^2+h^2) C_2(r) \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

for λ:



$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) g^2 C_2(G)\ \bar\sigma\cdot p$$

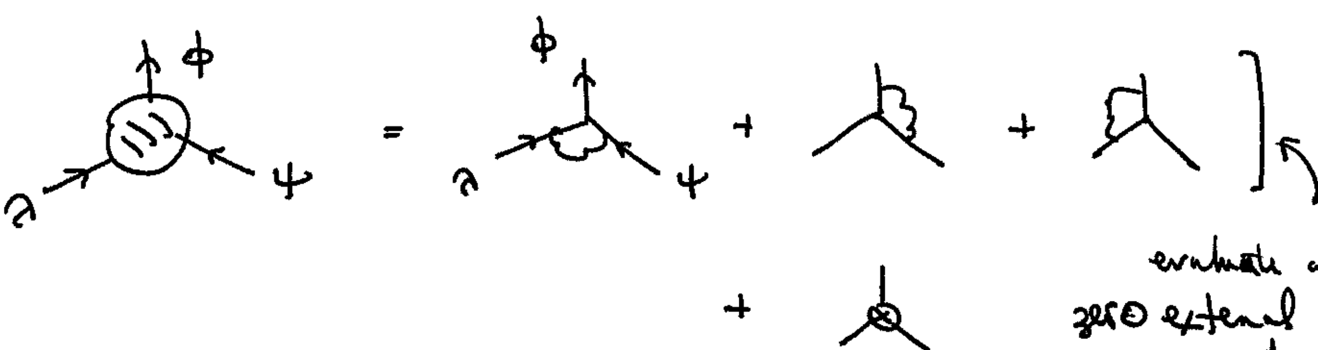


$$= \frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \cdot h^2 \cdot \underbrace{n_f\, C(r)}\ \bar\sigma\cdot p$$

$t^a t^a$ is replaced by $\mathrm{tr}[t^a t^b]$

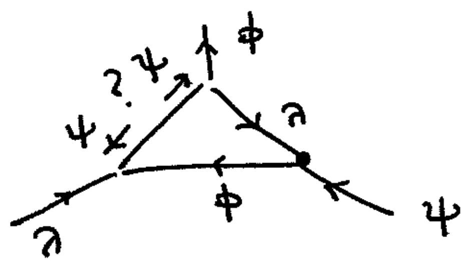
$$\delta_{Z\lambda} = -\left[g^2 C_2(G) + h^2 n_f\, C(r)\right] \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

Now look at the vertex diagrams. Start with h:

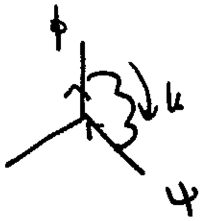


evaluate at zero external momentum.

Note that there is no h^3 diagram:



the arrows do not work out.



$$= (\sqrt{2}\, i h)(ig)^2 \int \frac{d^dk}{(2\pi)^d} (k^\mu t^b) \frac{i}{k^2} t^a \epsilon \frac{i\sigma\cdot k}{k^2} \bar\sigma_\mu t^b \cdot \frac{-i}{k^2}$$

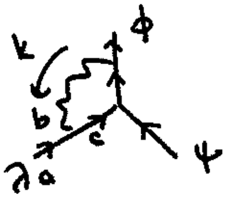
$$\overset{b}{\sim}\ \psi = igt^b$$

$$\overset{}{\sim}\ \lambda = igt^b_a = -gf^{abc}$$

$$= (\sqrt{2} h g^2)(+1)[t^b t^a t^b \epsilon] \int \frac{d^dk}{(2\pi)^d} \frac{\sigma\cdot k\ \bar\sigma\cdot k}{(k^2)^3}$$

$$= +\sqrt{2} h g^2 \frac{+i}{(4\pi)^2} \lg \Lambda^2/M^2 \cdot t^b t^a t^b \epsilon$$

$$(-1)^2 \underbrace{\overset{t_c}{\lambda^c} f^{cba} \lambda^a}\ \lambda^T \epsilon \psi$$

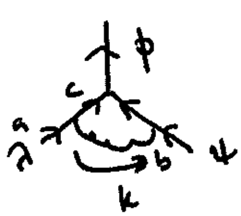


$$= \sqrt{2}\, i h \cdot ig(-g) \int \frac{d^dk}{(2\pi)^d} k^\mu t^b \frac{i}{k^2}$$

$$\cdot (\bar\sigma^\mu)^T \left(\frac{i\sigma\cdot k}{k^2}\right)^T \epsilon \quad f^{cba}\ t^c \quad \frac{-i}{k^2}$$

$$= +i\sqrt{2} h g^2 \int \frac{d^dk}{(2\pi)^d} (k\cdot\bar\sigma)^T(\sigma\cdot k)^T \frac{1}{(k^2)^3} \epsilon \quad f^{cba} t^b t^c$$

$$= \sqrt{2} h g^2 \frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \cdot \epsilon \cdot (+i f^{cba} t^b t^c)$$



$$= \sqrt{2}\, i h\, (-g)\ ig \int \frac{d^dk}{(2\pi)^d} \quad \frac{-i}{k^2}$$

$$(\bar\sigma^\mu)^T \left(\frac{i\sigma\cdot(-k)}{k^2}\right)^T \epsilon \quad \frac{i\sigma\cdot k}{k^2} \bar\sigma_\mu \quad f^{cba} t^c t^b$$

$$= \sqrt{2} h g^2 (-i) \cdot \int \frac{d^dk}{(2\pi)^d} \epsilon \frac{\sigma_\mu \bar\sigma\cdot k\ \sigma\cdot k\ \bar\sigma^\mu}{(k^2)^3} \cdot f^{cba} t^c t^b$$

$$= \sqrt{2}\, h g^2 \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \; \cdot 4\epsilon \cdot \left(-i f^{cba} t^c t^b\right)$$

now

$$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b]$$

$$= C_2(r)\, t^a + i f^{abc} t^b t^c$$

$$= C_2(r)\, t^a + i f^{abc} \tfrac{1}{2} i f^{bcd} t^d$$

$$= \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] t^a$$

$$-i f^{cba} t^c t^b = -i f^{cba} \tfrac{1}{2} i f^{cbd} t^d = +\tfrac{1}{2} C_2(G)\, t^a$$

$$+i f^{cba} t^b t^c = +\tfrac{1}{2} C_2(G)\, t^a$$

so

$$= \sqrt{2}\, h g^2 \cdot i \cdot \frac{1}{(4\pi)^2} \log \Lambda^2/M^2 \cdot \epsilon\, t^a$$

$$\cdot \left\{ + \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] + \tfrac{1}{2} C_2(G) + \tfrac{4}{2} C_2(G) \right\}$$

$$= \sqrt{2}\, i h\, g^2 \frac{\log \Lambda^2/M^2}{(4\pi)^2} \epsilon\, t^a$$

$$\cdot \left\{ + C_2(r) + 2\, C_2(G) \right\}$$

or $$\delta_h = -g^2 h \left(\frac{1}{(4\pi)^2} \lg \Lambda^2/M^2\right) [C_2(r) + 2C_2(G)]$$

then

$$\beta_h = M \frac{\partial}{\partial M} \left[-\delta_h + h/2 \left[\delta_{Z\lambda} + \delta_{Z\phi} + \delta_{Z\psi} \right] \right]$$

$$= M \frac{\partial}{\partial M} \Big[+ g^2 h \frac{1}{(4\pi)^2} [C_2(r) + 2C_2(G)] \lg \Lambda^2/M^2$$

$$+ h/2 \Big\{ - g^2 [C_2(G) + h^2 n_f C(r)] + (2g^2 - 2h^2) C_2(r)$$

$$- (g^2 + h^2) C_2(r) \Big\} \frac{1}{(4\pi)^2} \lg \Lambda^2/M^2 \Big]$$

$$= \frac{1}{(4\pi)^2} \Big\{ -2g^2 h [C_2(r) + 2C_2(G)]$$

$$+ h [g^2 C_2(G) + h^2 n_f C(r) + 2(h^2 - g^2) C_2(r)$$

$$+ (g^2 + h^2) C_2(r)] \Big\}$$

$$= + \frac{1}{(4\pi)^2} \Big\{ C_2(G) [-4g^2 h + g^2 h]$$

$$+ C_2(r) [-2g^2 h + 2h(h^2 - g^2) + h(g^2 + h^2)]$$

$$+ h^3 n_f C(r) \Big\}$$

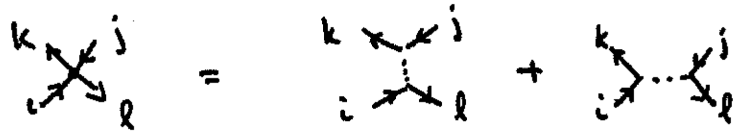
so

$$\beta_h = -\frac{1}{(4\pi)^2}\left\{ 3g^2 h\, C_2(G) - 3(h^2-g^2)h\, C_2(r) - h^3 n_f\, C(r) \right\}$$

$$= -\frac{1}{(4\pi)^2}\, g^3 \left[3C_2(G) - n_f C(r) \right] \quad \text{for } g = h$$

next, k:

to write the graphs, it is convenient to use the representation:



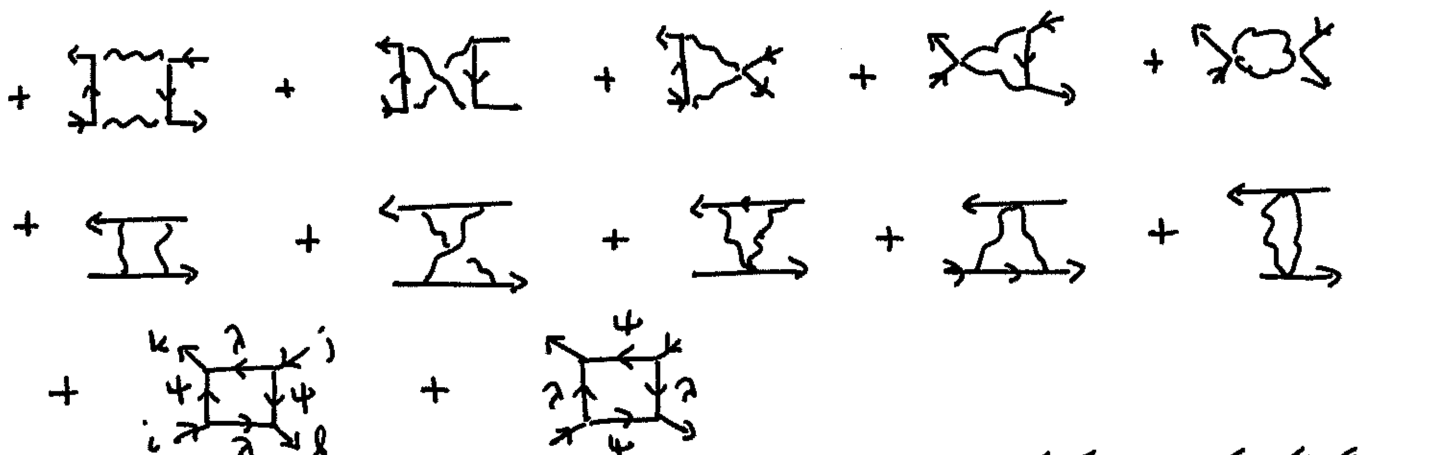
$$= -ik \cdot \left(t^a_{kj} \times t^a_{li} + t^a_{ki} \times t^a_{lj} \right)$$

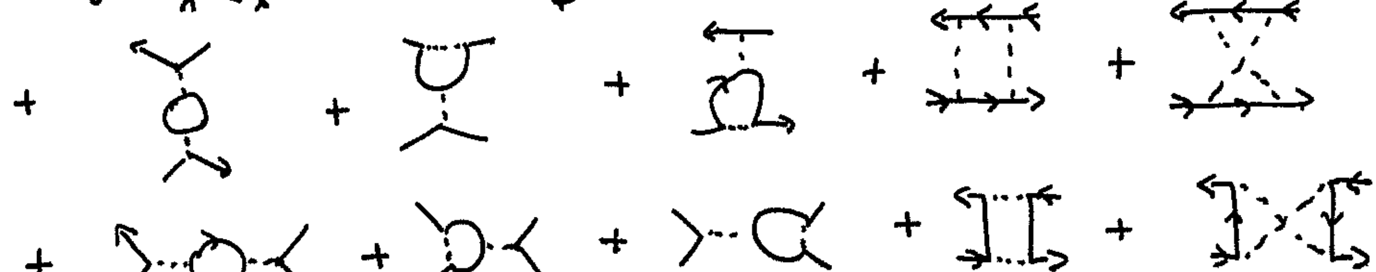
The vertex corrections are then:

[This is probably too many graphs to decently ask about on a take-home exam.]

 + (3 more)

 + (3 more)





$$= -i\kappa \cdot (ig)^2 \cdot \int \frac{d^dk}{(2\pi)^d} \; k_\mu \frac{i}{k^2} t^b \; t^a \; \frac{i}{k^2} t^b k^\mu$$

$$t^a \cdot \frac{-i}{k^2}$$

$$= -\kappa g^2 \cdot t^b t^a t^b \times -t^a \cdot \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2)^3}$$

$$= -\kappa g^2 \left[C_2(F) - \tfrac{1}{2} C_2(G) \right] (t^a \times t^a) \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

$$= -i\kappa \cdot (ig)^2 \int \frac{d^dk}{(2\pi)^d} \; \frac{-i}{k^2}$$

$$\cdot (t^a t^b) \; k_\mu \frac{i}{k^2} \times t^b t^a \; k_\mu \frac{i}{k^2}$$

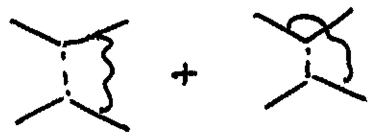
$$= -\kappa g^2 \left[(t^a t^b) \times (t^b t^a) \right] \cdot \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

$$= -i\kappa \; (ig)^2 \int \frac{d^dk}{(2\pi)^d} \; \frac{-i}{k^2}$$

$$\left[\frac{i}{k^2} (k_\mu t^b t^a) \right] \times \left(\frac{i}{k^2} (-k_\mu) t^b t^a \right)$$

$$= +\kappa g^2 \left[t^b t^a \otimes t^b t^a \right] \cdot \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

so



$$= -kg^2 \; [t^a, t^b] \times t^b t^a \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

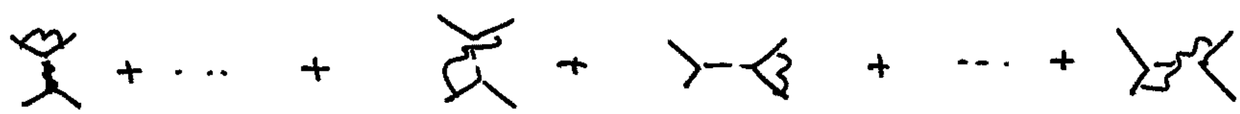
$$= kg^2 \; i f^{abc} \, (t^c \times t^b t^a) \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

$$= kg^2 \; \tfrac{1}{2} \, i f^{abc} \, i f^{bad} \; t^c \times t^d \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

$$= -kg^2 \; \tfrac{1}{2} C_2(G) \; (t^a \times t^a) \; \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

the other 3 diagrams of this type are equal to these.

the 6 diagrams in the 2nd line on p.19 are equal to these 6 with $k \leftrightarrow l$. So



$$= -\frac{i}{(4\pi)^2} \; 2kg^2 \; C_2(r) \left(t^a_{kj} \times t^a_{li} + t^a_{ki} \times t^a_{lj} \right) \log \Lambda^2/M^2$$

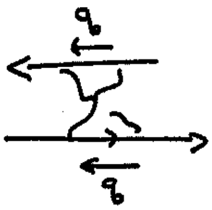
next, the graphs with gauge exchanges only:

$$= (ig)^4 \int \frac{d^dq}{(2\pi)^d} \left(q^\mu \frac{i}{q^2} q^\nu \right) \frac{-i}{q^2} \left(q_\nu \frac{i}{q^2} q_\mu \right) \frac{-i}{q^2}$$

$$\times (t^a t^b)_{kj} (t^b t^a)_{\ell i}$$

$$= + g^4 \int \frac{d^dq}{(2\pi)^d} \frac{(q^2)^2}{(q^2)^4} \; (t^a t^b)_{kj} \times (t^b t^a)_{\ell i}$$

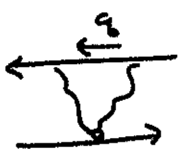
$$= \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \right) \cdot g^4 \; (t^a t^b)_{kj} \times (t^b t^a)_{\ell i}$$



$$= (ig)^4 \int \frac{d^dq}{(2\pi)^d} \left(q^\mu \frac{i}{q^2} q^\nu \right) \frac{-i}{q^2} \left((-q^\mu) \frac{i}{q^2} (-q^\nu) \right) \frac{-i}{q^2}$$

$$\times (t^a t^b)_{kj} \; (t^a t^b)_{\ell i}$$

$$= \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \right) g^4 \; (t^a t^b) \times (t^a t^b)$$



$$= (ig)^2 (ig^2) \int \frac{d^dq}{(2\pi)^d} \left(q^\mu \frac{i}{q^2} q^\nu \right) \left(\frac{-i}{q^2} \right)^2 g_{\mu\nu}$$

$$\times (t^a t^b)_{kj} \left(\{t^a, t^b\} \right)_{\ell i}$$

$$= (-g^4) \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \right) \; (t^a t^b) \times (\{t^a, t^b\})$$

$$= \; - g^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \left(\{t^a, t^b\} \right) \times (t^a t^b)$$

 $= \frac{1}{2}(ig^2)^2 \int \frac{d^d q}{(2\pi)^d} \left(\frac{-i}{q^2}\right)^2 g^{\mu\nu} g_{\mu\nu} (\{t^a, t^b\})(\{t^a, t^b\})$

$$= +g^4 \cdot \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2\right) \cdot 4 \cdot \frac{1}{2} (\{t^a, t^b\})(\{t^a, t^b\})$$

then

 +  +  +  + 

$$= \frac{i}{(4\pi)^2}(\lg \Lambda^2/M^2)\; g^4 \; (t^a t^b) \cdot (\{t^a, t^b\})$$

$$\cdot (1 - 2 + 4)$$

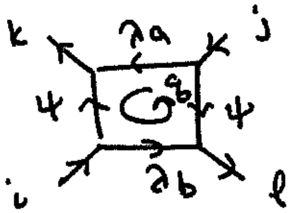
$$= \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2\right) (+3g^4) \, (t^a t^b)_{kj} \cdot (\{t^a, t^b\})_{li}$$

 +  +  +  + 

$$= \left(\frac{i}{(4\pi)^2} \lg \Lambda^2/M^2\right) (+3g^4) \quad (t^a t^b)_{ki} \cdot (\{t^a, t^b\})_{lj}$$

next, the graphs with fermion loops:

correct order! $\overbrace{\lambda\,\lambda^*}$



$$= \phi^*_k \lambda^{aT} t^a \epsilon \psi_k \; \psi^\dagger_i \epsilon \lambda^{b*} t^b \phi_i \; \phi_\ell \lambda^{Tb} t^b \psi_\ell \; \psi^\dagger_j \epsilon t^a \lambda^{*a} \phi_j$$

(-1) for wrong order

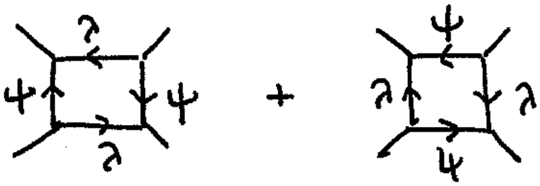
$$= (\sqrt{2} i h)(-i\sqrt{2} h)(\sqrt{2} i h)(-i\sqrt{2} h)\int \frac{d^dq}{(2\pi)^d}\, (t^a t^b)_{ki}\,(t^b t^a)_{\ell j}$$

$$\mathrm{tr}\left[\epsilon\, \frac{i\sigma\cdot(-q)}{q^2}\, \epsilon\, (-1)\left(\frac{i\sigma\cdot q}{q^2}\right)^T \epsilon \left(\frac{i\sigma\cdot(-q)}{q^2}\right) \epsilon \left(\frac{i\sigma\cdot q}{q^2}\right)^T\right]$$

$$= 4h^4\cdot(-1)\cdot\int\frac{d^dq}{(2\pi)^d}\, \mathrm{tr}\left(\frac{\sigma\cdot q\;\bar\sigma\cdot q\;\sigma\cdot q\;\bar\sigma\cdot q}{(q^2)^4}\right)(t^at^b)\times(t^bt^a)$$

$$= -8h^4\left(\frac{i}{(4\pi)^2}\log \Lambda^2/M^2\right)\,(t^at^b)\times(t^bt^a)$$

the other diagram gives the same result in the other channel.



$$= (-8h^4)\left(\frac{i}{(4\pi)^2}\log \Lambda^2/M^2\right)\left[(t^at^b)_{kj}\times(t^bt^a)_{\ell i} + (t^at^b)_{ki}\times(t^bt^a)_{\ell j}\right]$$

Finally, the diagrams with 2 scalar vertices:

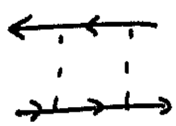
$$= (-ik)^2 \int \frac{d^dq}{(2\pi)^d} \left(\frac{i}{q^2}\right)^2 (t^a)_{kj} \cdot \mathrm{tr}(t^a t^b) \cdot (t^b)_{li}$$

$$= k^2 \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) \quad t^a_{kj}\, t^a_{li} \cdot n_f C(r)$$

$$= (-ik)^2 \int \frac{d^dq}{(2\pi)^d} \left(\frac{i}{q^2}\right)^2 (t^b t^a t^b)_{kj} \cdot (t^a)_{li}$$

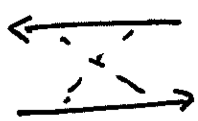
$$= k^2 \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] t^a_{kj}\, t^a_{li}$$

$$= k^2 \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) \left[C_2(r) - \tfrac{1}{2} C_2(G)\right] t^a_{kj}\, t^a_{li}$$



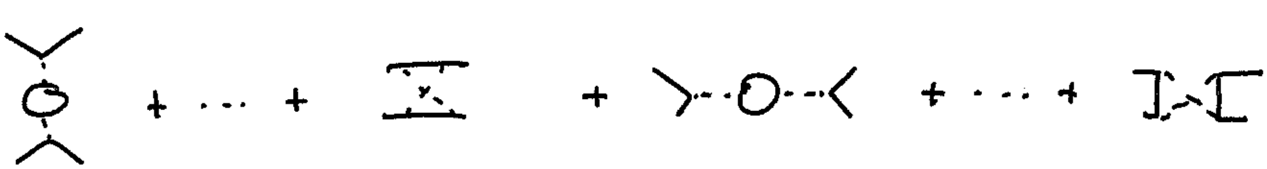
$$= (-ik)^2 \int \frac{d^dq}{(2\pi)^d} \left(\frac{i}{q^2}\right) \quad (t^a t^b)_{kj}\, (t^b t^a)_{li}$$

$$= k^2 \frac{i}{(4\pi)^2} (\log \Lambda^2/M^2) \quad (t^a t^b)_{kj}\, (t^b t^a)_{li}$$



$$= k^2 \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right) \quad (t^a t^b)_{kj}\, (t^a t^b)_{li}$$

the five diagrams in the last line of p. 19 give the same result in the other channel.



$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \cdot k^2 \cdot$$

$$\Big\{ \left[(t^a)_{kj} (t^a)_{li} + (t^a)_{ki} (t^a)_{lj} \right] \left[2C_2(r) - C_2(G) + n_f C(r) \right]$$

$$+ \quad (t^a t^b)_{kj} (\{t^a_j t^b\})_{li} + (t^a t^b)_{ki} (\{t^a_j t^b\})_{lj} \Big\}$$

Now add all of the contributions:

$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right)$$

$$\cdot \Big[\; - 2 g^2 k \, C_2(r) \; \left[t^a \cdot t^a + t^a \cdot t^a \right]$$

$$+ \; 3 g^4 \; \left[(t^a t^b) \cdot (\{t^a_j t^b\}) + (t^a t^b) \cdot (\{t^a_j t^b\}) \right]$$

$$- \; 8 h^4 \; \left[(t^a t^b) \cdot (t^b t^a) + (t^a t^b) (t^b t^a) \right]$$

$$+ \; k^2 \; \left[2 C_2(r) - C_2(G) + n_f C(r) \right]$$

$$+ \; k^2 \; \left[(t^a t^b) (\{t^a_j t^b\}) + (t^a t^b) (\{t^a_j t^b\}) \right] \; \Big]$$

this is a very complicated expression in the general case. However, it has a nice simplification for the supersymmetric situation: $h = g$ $k = g^2$. From here on, I will restrict myself to that case.

$$= \frac{i}{(4\pi)^2}\left(\log \Lambda^2/M^2\right)$$

$$\cdot \Big[\left(t^a \cdot t^a + t^a \cdot t^a\right) \cdot \Big[-\cancel{2g^4 C_2(r)} + g^4 \left[2C_2(r) - C_2(G) + n_f C(r) \right] \Big]$$

$$+ \left[(t^a t^b) \cdot (t^a t^b + t^b t^a) + (t^a t^b) \cdot (t^a t^b + t^b t^a) \right] \cdot \left[3g^4 + g^4 \right]$$

$$+ \left[(t^a t^b)\,(t^b t^a) + (t^a t^b)\,(t^b t^a) \right] \left[-8g^4 \right] \Big]$$

$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) g^4$$

$$\Big[\left(t^a \cdot t^a + t^a \cdot t^a\right) \left[-C_2(G) + n_f C(r) \right]$$

$$+ 4 \left[(t^a t^b)(t^a t^b - t^b t^a) + (t^a t^b)(t^a t^b - t^b t^a) \right] \Big]$$

now $\quad t^a t^b \, [t^a, t^b] \; = \; i f^{abc} \, (t^a t^b)\,(t^c)$

$$= i f^{abc} \cdot \tfrac{1}{2} i f^{abd} (t^d)(t^c) \; = \; -\tfrac{1}{2} C_2(G) \; t^a \cdot t^a$$

then

 $= \dfrac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \quad (t^a \cdot t^a + t^a \cdot t^a)$

$$\cdot \left[- C_2(G) + n_f C(r) + 4\left(-\tfrac{1}{2} C_2(G)\right)\right]$$

$$= \frac{i}{(4\pi)^2} \lg \Lambda^2/M^2 \quad (t^a \cdot t^a + t^a \cdot t^a)\left[-3 C_2(G) + n_f C(r)\right]$$

this is cancelled by

 $= -i \, \delta_k \; [t^a \cdot t^a + t^a \cdot t^a]$

so $\quad \delta_k = \left(\dfrac{-1}{(4\pi)^2} \lg \Lambda^2/M^2\right)\left[3 C_2(G) \;\; n_f C(r)\right]$

from p. 13, for $g = h$, $\quad \delta_{Z\phi} = 0$

then

$$\beta_k = M \frac{\partial}{\partial M}\left[-\delta_k + 4 \cdot \tfrac{1}{2} \delta_{Z\phi}\right]$$

$$= \frac{-2}{(4\pi)^2}\left[3 \, C_2(G) - n_f C(r)\right] g^4$$

we have now seen that, for $g = h$ $k = g^2$:

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right]$$

$$\beta_h = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right] = \beta_g$$

$$\beta_k = -2\frac{g^4}{(4\pi)^2}\left[3C_2(G) - n_f C(r)\right] = 2g\,\beta_g$$

This is just what is needed for the relations

$$h = g \qquad k = g^2$$

to be preserved by the renormalization group.

e.) To compute the QCD contributions to the β functions of the three fermion-fermion-boson interactions in (5) we need the QCD corrections to these vertices and the QCD contributions to $\delta Z\phi$ $\delta Z\psi$:

From pp. 13, 14

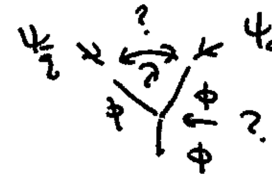
$$\delta Z\phi_q = 0 \qquad \delta Z\psi_q = -2g^2 C_2(r)\frac{1}{(4\pi)^2}\log\frac{\Lambda^2}{M^2}$$

for $q, \bar{q}$

of course, for h: $\underline{\delta Z\phi_h = \delta Z\psi_h = 0}$ since h is a color singlet.

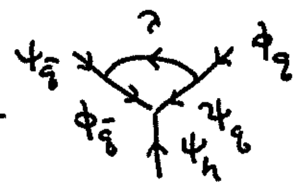
for the vertex $\qquad \delta\mathcal{L} = y\, \psi_{\bar{q}}\, \epsilon\, \psi_q\, \phi_n$

the only correction is 

since there is no possible diagram with ∂ : 

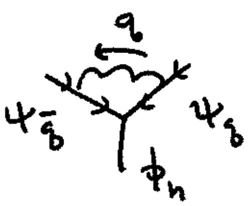
However, for $\qquad \delta\mathcal{L} = y\, \psi_{\bar{q}}\, \epsilon\, \phi_q\, \psi_n$

there are corrections 

$+$ 

The third vertex $\quad \delta\mathcal{L} = y\, \phi_{\bar{q}}\, \psi_q\, \epsilon\, \psi_n \quad$ has the same situation as this.

So, let's compute these three diagrams:



$$\underbrace{\psi^\dagger_{\bar{q}}\, \bar{\sigma}^\mu t^a_{\bar{r}}\, \psi_{\bar{q}}}_{(-1)^2} \quad \psi^T_{\bar{q}}\, \epsilon\, \psi_q\, \phi_n \quad \psi^\dagger_q\, \bar{\sigma}_\mu t^a \psi_q$$

$$= (iy)(ig)^2 \int \frac{d^dq}{(2\pi)^d}\, (\bar{\sigma}^\mu)^T (t^a_{\bar{r}})^T \left(\frac{i\sigma\cdot q}{q^2}\right)^T \epsilon\, \frac{i\sigma\cdot(-q)}{q^2}\, \bar{\sigma}_\mu\, t^a_r \left(\frac{-i}{q^2}\right)$$

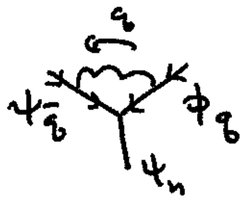
$$= -\, y g^2\, (t^{aT}_{\bar{r}} t^a_r) \int \frac{d^dq}{(2\pi)^d}\, \frac{\sigma^\mu\, \bar{\sigma}\cdot q\, \sigma\cdot q\, \bar{\sigma}_\mu}{(q^2)^3}$$

$$= -\, y g^2\, [(t^a_{\bar{r}})^T t^a_r] \int \frac{d^dq}{(2\pi)^d}\; 4\, \frac{1}{(q^2)^2}$$

now $\quad t^a_{\bar r} = -(t^a_r)^T \qquad (t^a_{\bar r})^T t^a_r = - t^a_r t^a_r = -C_2(r)$

so

$$= +4C_2(r)\, y g^2 \epsilon \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right)$$

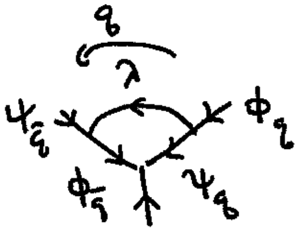


$$= (iy)(ig)^2 \int \frac{d^dq}{(2\pi)^d} (\bar\sigma^\mu)^T (t^a_{\bar r})^T \left(\frac{i\sigma\cdot q}{q^2}\right)^T \epsilon$$

$$\left(\frac{+i}{q^2}\right)(-q_\mu)\, t^a_r \left(\frac{-i}{q^2}\right)$$

$$= -y g^2 (t^a_{\bar r})^T t^a_r\, \epsilon \int \frac{d^dq}{(2\pi)^d} \frac{\sigma\cdot q\; \bar\sigma\cdot q}{(q^2)^3}$$

$$= +y g^2 C_2(r)\, \epsilon \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2\right)$$



$$(\psi_{\bar q}^T \epsilon (-t^a_{\bar r}) \lambda^a \phi^*_{\bar q})(\phi_{\bar q}\, \psi_q^T \epsilon\, \psi_h)(\psi_q^\dagger \epsilon\, \lambda^{*a} t^a_r \phi_q)$$

$$(-1)^2$$

$$= (\sqrt{2} i g)(iy)(-\sqrt{2} i g) \int \frac{d^dq}{(2\pi)^d} (-t^a_r t^a_r) \frac{i}{q^2}$$

$$\cdot\, \epsilon \frac{i\sigma\cdot q}{q^2} \epsilon^T \left(\frac{i\sigma\cdot(-q)}{q^2}\right)^T \epsilon \qquad\qquad \epsilon^T = -\epsilon$$

$$\epsilon^2 = -1$$

$$= 2 y g^2 C_2(r) \int \frac{d^dq}{(2\pi)^d} \epsilon \left(\frac{\sigma q}{q^2} \frac{\bar\sigma\cdot q}{q^2} \frac{1}{q^2}\right)$$

$$= 2 y g^2 C_2(r) \frac{i}{(4\pi)^2} \log \Lambda^2/M^2$$

cancelling the corrections with  : $i\,\delta_y\,\epsilon$

$$\delta_y(\psi_{\bar q}\psi_q\phi_n) = -4yg^2\frac{C_2(r)}{(4\pi)^2}\log\Lambda^2/M^2$$

$$\delta_y(\psi_{\bar q}\phi_q\psi_n) = -3yg^2\frac{C_2(r)}{(4\pi)^2}\log\Lambda^2/M^2$$

then the QCD contributions to the β functions are:

$$\beta_y(\psi_{\bar q}\psi_q\phi_n) = M\frac{\partial}{\partial M}\left[-\delta_y(\psi_{\bar q}\psi_q\phi_n) + \frac{2}{2}y\,\delta_{Z\psi_q}\right]$$

$$= -8yg^2C_2(r)\frac{1}{(4\pi)^2} + y\,4g^2C_2(r)\frac{1}{(4\pi)^2}$$

$$= -4\frac{yg^2C(r)}{(4\pi)^2}$$

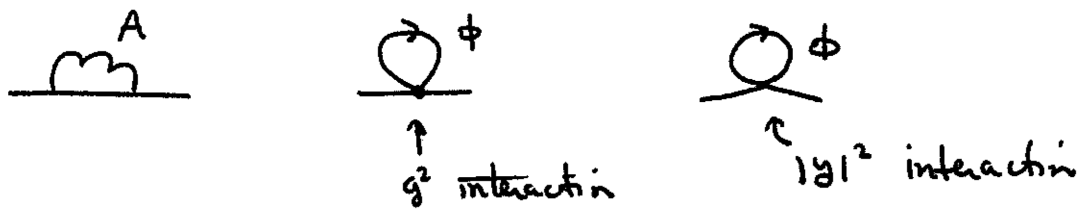
$$\beta_y(\psi_{\bar q}\phi_q\psi_n) = M\frac{\partial}{\partial M}\left[-\delta_y(\psi_{\bar q}\phi_q\psi_n) + \frac{1}{2}y(\delta_{Z\psi_q} + \delta_{Z\phi_q})\right]$$

$$= -6yg^2C_2(r)\frac{1}{(4\pi)^2} + y(2+0)g^2\frac{C_2(r)}{(4\pi)^2}$$

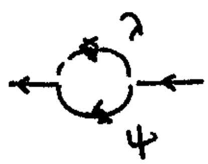
$$= -4\frac{yg^2C(r)}{(4\pi)^2}$$

these β functions are <u>equal</u>!

f.) The corrections to the scalar masses come from the diagrams on p. 5 and p. 10. The supersymmetric contributions cancel, but if there are supersymmetry-breaking perturbations such as eq. (6), we can get a nonzero contribution. The scalar masses potentially contribute to



There is also a contribution from



if the λ are massive.

I would first like to show that the contribution from the g^2 diagrams cancel. Expand in the ϕ mass

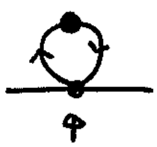
$$\frac{i}{q^2-m^2} = \frac{i}{q^2} + \frac{i}{q^2}(-im^2)\,\frac{i}{q^2} + \cdots$$

The $g^2 m^2$ terms are then: (at zero external momentum)

$$= (ig)^2 \int \frac{d^dq}{(2\pi)^d}\, t^a q^\mu \frac{i}{q^2} (-im^2) \frac{i}{q^2} q_\mu t^a \frac{-i}{q^2}$$

$$= - g^2 C_2(r) \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2)^3}$$

$$= -i\, g^2 C_2(r) \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$



g^2

$$= -ik \int \frac{d^dq}{(2\pi)^d}\, t^a \frac{i}{q^2} (-im^2) \frac{i}{q^2} t^a$$

$$= + g^2 C_2(r) \int \frac{d^dq}{(2\pi)^d} \left(\frac{1}{q^2}\right)^2 \qquad \text{using } k = g^2$$

$$= + i g^2 C_2(r) \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

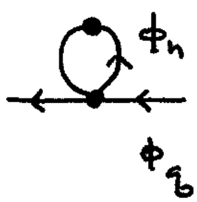
[Note: this cancellation is specifically in Feynman gauge; otherwise the cancellation involves these two diagrams and δZ_ϕ.]

The remaining contributions come from the interaction

$$\delta\mathcal{L} = - y^2 |\phi\phi|^2$$

In the specific model of eq. (5), there are three vertices of this type, and these contribute to the ϕ_h, ϕ_q, and $\phi_{\bar{q}}$ masses. Let's begin by working out one of these contributions quite explicitly:

the contribution to  ϕ_q from $\delta\mathcal{L} = -y\,|\phi_q\phi_h|^2$



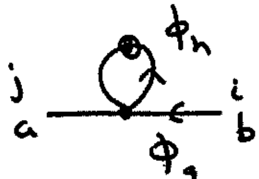
$$= (-iy^2)\int\frac{d^dq}{(2\pi)^d}\,\frac{i}{q^2}\,(-iM_h^2)\,\frac{i}{q^2}$$

$$= y^2\left(\frac{i}{(4\pi)^2}\log\Lambda^2/M^2\right)\cdot M_h^2$$

There is also a flow of indices. Let $i,j = 123$ be color indices, $a,b = 1,2$ be $SU(2)$ indices. The vertex is, explicitly

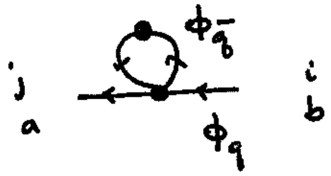
$$\delta\mathcal{L} = -y^2\,\phi_h^{a*}\,\phi_{qa}^{*i}\,\phi_{qi}^{b}\,\phi_{hb}$$

contracting ϕ_h^*, ϕ_h gives δ_{ab} so

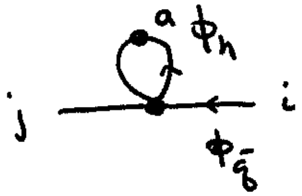


$$= \left(\frac{i}{(4\pi)^2}\log\Lambda^2/M^2\right)(y^2M_h^2)\;\delta_{ij}\delta_{ab}$$

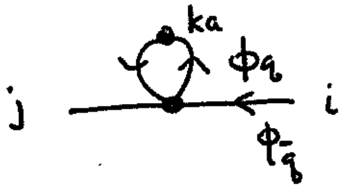
similarly :



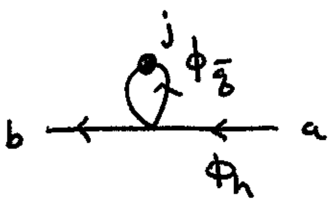
$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \quad y^2 M_{\bar{q}}^2 \quad \delta_{ij} \delta_{ab}$$



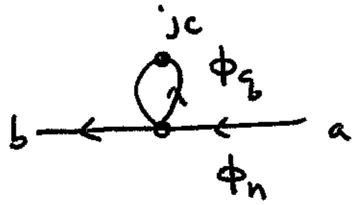
$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \quad y^2 M_h^2 \quad \delta_{ij} \, (\delta_{aa})$$



$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \quad y^2 M_q^2 \quad \delta_{ij} (\delta_{aa})$$



$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \quad y^2 M_{\bar{q}}^2 \, (\delta_{jj}) \, \delta_{ab}$$



$$= \left(\frac{i}{(4\pi)^2} \log \Lambda^2/M^2 \right) \quad y^2 M_q^2 \, (\delta_{jj}) \, \delta_{ab}$$

$$\delta_{aa} = 2 \qquad \delta_{jj} = 3$$

from p. 6, there is also a δz_ϕ proportional to y^2:

$$\delta z_{\phi_q} = -2 y^2 \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

$$\delta z_{\phi_{\bar{q}}} = -2 y^2 \frac{1}{(4\pi)^2} \log \Lambda^2/M^2 \cdot (\delta_{aa})$$

$$\delta z_{\phi_h} = -2 y^2 \frac{1}{(4\pi)^2} (\log \Lambda^2/M^2) \cdot (\delta_{jj})$$

the contributions to  on p. 36 are cancelled by $-i\,\delta_{M^2}$, so

$$\delta_{M_q^2} = \left(\frac{1}{(4\pi)^2}\log \Lambda^2/M^2\right)\cdot y^2\left(M_{\bar q}^2 + M_h^2\right)$$

$$\delta_{M_{\bar q}^2} = \left(\frac{1}{(4\pi)^2}\log \Lambda^2/M^2\right)\cdot y^2\left(M_q^2 + M_h^2\right)\cdot 2$$

$$\delta_{M_h^2} = \left(\frac{1}{(4\pi)^2}\log \Lambda^2/M^2\right)\cdot y^2\left(M_q^2 + M_{\bar q}^2\right)\cdot 3$$

then

$$\beta(M_q^2) = M\frac{\partial}{\partial M}\left(-\delta_{M_q^2} + \frac{2}{2} M_q^2\, \delta_{Z\phi_q}\right)$$

$$= +\frac{2}{(4\pi)^2}\, y^2\left(M_{\bar q}^2 + M_h^2\right) + M_q^2\, y^2 \frac{2}{(4\pi)^2}$$

$$= \frac{2}{(4\pi)^2}\, y^2\left(M_{\bar q}^2 + M_h^2 + M_q^2\right)$$

$$\beta(M_{\bar q}^2) = \frac{2}{(4\pi)^2}\, y^2\left(M_q^2 + M_h^2 + M_{\bar q}^2\right)\cdot 2$$

$$\beta(M_h^2) = \frac{2}{(4\pi)^2}\, y^2\left(M_q^2 + M_{\bar q}^2 + M_h^2\right)\cdot 3$$

so

$$\frac{d}{d\log Q} M_{\tilde{q}}^2 = + \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_{\tilde{q}}^2 + M_{\tilde{\bar{q}}}^2 \right)$$

$$\frac{d}{d\log Q} M_{\tilde{\bar{q}}}^2 = + 2 \cdot \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_{\tilde{q}}^2 + M_{\tilde{\bar{q}}}^2 \right)$$

$$\frac{d}{d\log Q} M_h^2 = + 3 \cdot \frac{2}{(4\pi)^2} y^2 \left(M_h^2 + M_{\tilde{q}}^2 + M_{\tilde{\bar{q}}}^2 \right)$$

the sign is such that the M^2 increase at large Q, decrease at small Q.

g.) The qualitative nature of the theory depends on the initial conditions on the RG flow. But

assume ~~that~~ $M_h^2 = M_{\tilde{q}}^2 = M_{\tilde{\bar{q}}}^2 > 0$ at a high scale Q_0

Then, at smaller Q, $M_h^2 \to 0$ faster than $M_{\tilde{q}}^2, M_{\tilde{\bar{q}}}^2$. Going further M_h^2 becomes negative, Φ_h acquires a VEV, and $SU(2)$ is spontaneously broken. This is a concrete mechanism for $SU(2) \times U(1)$ electroweak symmetry breaking.