# Physics 332 – Final Exam

## Solutions

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1.) If we only consider the top quark effect on the running of $\alpha_s$, we have, as in QED

$$g^{-2}(Q) = g^{-2}(M)\left(1 - [\Pi(Q) - \Pi(M)]\right)$$

where
$$\Pi(Q) - \Pi(M) = -\frac{2\alpha_s}{\pi} \cdot \underbrace{\frac{1}{2}}_{C(r)} \cdot \int_0^1 dx\, x(1-x) \log\left(\frac{m_t^2 - x(1-x)M^2}{m_t^2 - x(1-x)Q^2}\right)$$

for $Q^2 \gg m_t^2$

$$\Pi(Q) - \Pi(M) = +\frac{\alpha_s}{\pi} \cdot \frac{1}{6} \log Q^2/M^2$$

$$\alpha_s^{-1}(Q) = \alpha_s^{-1}(M) - \frac{1}{6\pi} \log Q^2/M^2$$

with all 6 quarks and the gluons, we have $\quad Q^2 \gg m_t^2$

$$\alpha_s^{-1}(Q) = \alpha_s^{-1}(M) + \frac{1}{4\pi} b_{0,6} \log Q^2/M^2$$

$$b_{0,n_f} = \left(11 - \frac{2}{3} \cdot n_f\right)$$

Then the formula for $\alpha_s^{-1}(Q)$, with full dependence on the top quark mass, should be

$$\alpha_s^{-1}(Q) = \alpha_s^{-1}(M) + \frac{b_{0S}}{4\pi} \log Q^2/M^2 + \frac{1}{\pi}\int_0^1 dx\, x(1-x) \log\left(\frac{m_t^2 - x(1-x)M^2}{m_t^2 - x(1-x)Q^2}\right)$$

work with $Q^2 < 0$ $\quad Q^2 = -|Q^2|$

for $|Q^2| \gg m_t^2$

$$\int dx\, x(1-x) \log(m_t^2 + x(1-x)|Q^2|)$$

$$= \int dx\, x(1-x)\left[\log|Q^2| + \log x(1-x) + \mathcal{O}(m_t^2/Q^2)\right]$$

$$= \frac{1}{6}\log|Q^2| - \frac{5}{18} + \mathcal{O}(m_t^2/Q^2)$$

from
$$\int_0^1 dx\, x(1-x) \log x(1-x) = 2\int dx\, x(1-x)\log x$$

$$= 2\int_0^1 dx\, x\log x - 2\int_0^1 dx\, x^2 \log x$$

$$= 2\left(-\frac{1}{4} + \frac{1}{9}\right)$$

$$= \frac{1}{6}\left(\log|Q^2| - \frac{5}{3}\right)$$

for $|Q^2| \ll m_t^2$

$$\int dx\, x(1-x) \log(m_t^2 + x(1-x)|Q^2|) = \int dx\, x(1-x)\left[\log m_t^2 + \mathcal{O}\left(\frac{Q^2}{m_t^2}\right)\right]$$

$$= \frac{1}{6}\log m_t^2 + \cdots$$

so

$$\alpha_s^{-1}(Q) = \begin{cases} C + \frac{b_{05}}{4\pi} \lg|Q^2| - \frac{1}{6\pi}\left(\lg Q^2 - \frac{5}{3}\right) & |Q| \gg m_t^2 \\ C + \frac{b_{05}}{4\pi} \lg|Q^2| - \frac{1}{6\pi} \lg m_t^2 & |Q^2| \ll m_t^2 \end{cases}$$

for $|Q|^2 \gg m_t^2$

$$\alpha_s^{-1}(Q) = \frac{b_{06}}{4\pi} \lg|Q^2| + \left(C + \frac{5}{18\pi}\right)$$

$$\equiv \frac{b_{06}}{4\pi} \lg |Q^2|/\Lambda_6^2$$

so that $\log \Lambda_6^2 = -\frac{4\pi}{b_{06}}\left(C + \frac{5}{18\pi}\right)$

or $$C + \frac{5}{18\pi} = -\frac{b_{06}}{4\pi} \lg \Lambda_6^2$$

then for $|Q^2| \ll m_t^2$

$$\alpha_s^{-1}(Q) = \frac{b_{05}}{4\pi} \lg|Q^2| - \frac{b_{06}}{4\pi} \lg\Lambda_6^2 - \frac{5}{18\pi} - \frac{1}{6\pi} \lg m_t^2$$

$$\equiv \frac{b_{05}}{4\pi} \lg |Q^2|/\Lambda_5^2$$

where

$$\frac{b_{05}}{4\pi} \lg \Lambda_5^2 = \frac{b_{06}}{4\pi} \lg \Lambda_6^2 + \frac{1}{6\pi} \lg m_t^2 + \frac{5}{18\pi}$$

$$\lg \Lambda_5 = \frac{b_{06}}{b_{05}} \lg \Lambda_6 + \frac{2/3}{b_{05}} \lg m_t + \frac{5/9}{b_{05}}$$

$$\Lambda_5 = (\Lambda_6)^{b_{06}/b_{05}} (m_t)^{(2/3)/b_{05}} e^{\frac{5}{9b_{05}}}$$

Similarly

$$\Lambda_4 = (\Lambda_5)^{b_{05}/b_{04}} (m_b)^{2/3/b_{04}} e^{\frac{5}{9b_{04}}}$$

$$\Lambda_3 = (\Lambda_4)^{b_{04}/b_{03}} (m_c)^{2/3/b_{03}} e^{\frac{5}{9b_{03}}}$$

now

$$b_{06} = \frac{21}{3} \qquad b_{05} = \frac{23}{3} \qquad b_{04} = \frac{25}{3} \qquad b_{03} = \frac{27}{3}$$

$$\Lambda_5 = (\Lambda_6)^{21/23} (m_t)^{2/23} e^{5/69}$$

$$\Lambda_4 = (\Lambda_6)^{21/25} (m_t)^{2/25} (m_b)^{2/25} e^{10/75}$$

$$\Lambda_3 = (\Lambda_6)^{21/27} (m_t m_b m_c)^{2/27} e^{15/81}$$

we'll need later

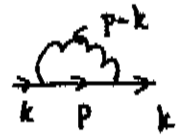
$$\Lambda_4 = (\Lambda_5)^{23/25} (m_b)^{2/25} e^{5/75}$$

2.) Let  $= -i\Sigma(\not k)$

then the pole in the top quark propagator is located at

$$\not k - m_t - \Sigma(\not k) = 0 \qquad \not k = m_{t,\text{pole}}$$

$$m_{t\,\text{pole}} = m_t + \Sigma(m_{t\,\text{pole}})$$
$$\cong m_t + \Sigma(m_t) \qquad \text{if } \Sigma(m_t) \text{ is } O(\alpha_s)$$

$\Sigma(\not k) = i$ 

$$= i\,(ig)^2 t^a t^a \int \frac{d^d p}{(2\pi)^d} \frac{-i}{(p-k)^2}\, \gamma_\mu \frac{i(\not p + m_t)}{p^2 - m_t^2} \gamma^\mu$$

$$= -ig^2 \cdot \frac{4}{3} \cdot \int_0^1 dx \int \frac{d^d \mathbb{P}}{(2\pi)^d} \frac{\gamma_\mu (\not p + m_t) \gamma^\mu}{[\mathbb{P}^2 + x(1-x)k^2 - (1-x)m_t^2]^2}$$

where $\quad \mathbb{P} = p - xk \qquad p = \mathbb{P} + xk$

$$\gamma^\mu (\not p + m_t)\gamma_\mu = (2-d)\not p + d\, m_t$$
$$= (-2+\varepsilon)\not p + (4-\varepsilon) m_t$$
$$= (4m_t - 2\not p) + \varepsilon(-m_t + \not p)$$
$$= (\text{linear in } \mathbb{P}) + (4m_t - 2x\not k) + \varepsilon(-m_t + x\not k)$$

$$= -ig^2 \cdot \frac{4}{3} \int_0^1 dx \quad \frac{i}{(4\pi)^{d_2}} \Gamma(2-d_2) \frac{1}{[(1-x)m_t^2 - x(1-x)k^2]^{2-d_2}}$$

$$\cdot \{(4m_t - 2x\not k) - \varepsilon(m_t - x\not k)\}$$

$$\Sigma(\not{k}) = \frac{4}{3} g^2 \int_0^1 dx \, \frac{1}{(4\pi)^2} \left[ \left( \frac{2}{\epsilon} - \gamma + \log 4\pi \right) - \log\left[(1-x)m_t^2 - x(1-x)k^2\right] \right]$$

$$\cdot \left[ (4m_t - 2x\not{k}) - \varepsilon\,(m_t - x\not{k}) \right]$$

apply the $\overline{MS}$ subtraction and letting $\varepsilon \to 0$

$$\Sigma(\not{k})\Big|_{\overline{MS},M} = \frac{4}{3}\frac{g^2}{(4\pi)^2} \int_0^1 dx \left\{ \log\left[ \frac{M^2}{(1-x)m_t^2 - x(1-x)k^2} \right] (4m_t - 2x\not{k}) - 2(m_t - x\not{k}) \right\}$$

now set $M = m_t$ $\quad k^2 = m_t^2$ $\quad \not{k} = m_t$

$$\Sigma(m_t)\Big|_{\overline{MS}\text{ at } m_t} = \frac{4}{3}\frac{g^2}{(4\pi)^2} \int_0^1 dx \left\{ \log\left[\frac{1}{(1-x)^2}\right] (4m_t - 2x m_t) - 2(m_t - x m_t) \right\}$$

$$= \frac{4}{3}\frac{\alpha_s}{4\pi} m_t \left\{ \int_0^1 dx \, [-2\log(1-x)]\,[4-2x] - 2\left(1 - \tfrac{1}{2}\right) \right\}$$

$$\int dx \log(1-x)\,(4-2x) = \int_0^1 dy \log y \,(2+2y)$$

$$= 2\left(-1 - \tfrac{1}{4}\right) = -\frac{5}{2}$$

$$= \frac{4}{3}\frac{\alpha_s}{4\pi} m_t \cdot \left[ (-2)\left(-\tfrac{5}{2}\right) - 2\left(\tfrac{1}{2}\right) \right]$$

$$= \frac{4}{3}\frac{\alpha_s}{\pi} m_t$$

so if $m_t$ is defined by $\overline{MS}$ subtraction at $M = m_t$, the pole in the t quark propagator is located at

$$m_{t,\text{pole}} = m_t \left(1 + \frac{4\alpha_s}{3\pi}\right)$$

using $m_{t,\text{pole}} = 175.$ GeV $\quad \alpha_s(m_t) = 0.11 \quad \frac{4\alpha_s}{3\pi} = 4.7\%$

$$m_t(m_t)_{\overline{MS}} = 167. \text{ GeV}$$

3.) To scale quark masses from one value of Q to another, use

$$\frac{\overline{m}(Q)}{\overline{m}(M)} = \left[\frac{\log(M/\Lambda)}{\log(Q/\Lambda)}\right]^{4/b_0}$$

Evaluate this formula for the transitions between quark thresholds.

$|\leftrightarrow|\longleftrightarrow|\longleftrightarrow|$

$m_c \quad$ 2 GeV $\approx 2m_c \quad 2m_b \quad$ reference point at $m_t|_{\overline{MS}}$

$\uparrow$ $n_f = 4$ $\qquad \uparrow$ $n_f = 5$

First, we need the $\Lambda$'s and $b_0$'s.

$n_f = 5 \qquad b_0 = \frac{23}{3}$

$$\alpha_s(m_t) = 0.11 = \frac{2\pi}{b_0 \lg m_t/\Lambda_5} \Rightarrow \Lambda_5 = 97 \text{ MeV}$$

from p. 4 $\qquad n_f = 4 \rightarrow b_0 = \frac{25}{3}$

$$\Lambda_4 = 140 \text{ MeV}$$

then

$$\left(\frac{\log[2m_b/\Lambda_4]}{\log[2\text{GeV}/\Lambda_4]}\right)^{4\cdot\frac{2}{25}} = 1.2$$

$$\left(\frac{\log[m_t/\Lambda_5]}{\log[2m_b/\Lambda_5]}\right)^{4\cdot\frac{3}{23}} = 1.3$$

$$\left[\frac{\log(2m_b/\Lambda_4)}{\log(m_c/\Lambda_4)}\right]^{4\cdot\frac{3}{25}} = 1.4$$

so

$$m_u(m_t) = (3\text{ MeV})/(1.2)\cdot(1.3) = 2\text{ MeV}$$

$$m_d(m_t) = (5\text{ MeV})/(1.2)\cdot(1.3) = 3\text{ MeV}$$

$$m_s(m_t) = (100\text{ MeV})/(1.2)(1.3) = 62\text{ MeV}$$

$$m_c(m_t) = (1.2\text{ GeV})/(1.4)(1.3) = 670\text{ MeV}$$

$$m_b(m_t) = (4.2\text{ GeV})/1.4 = 2900\text{ MeV}$$

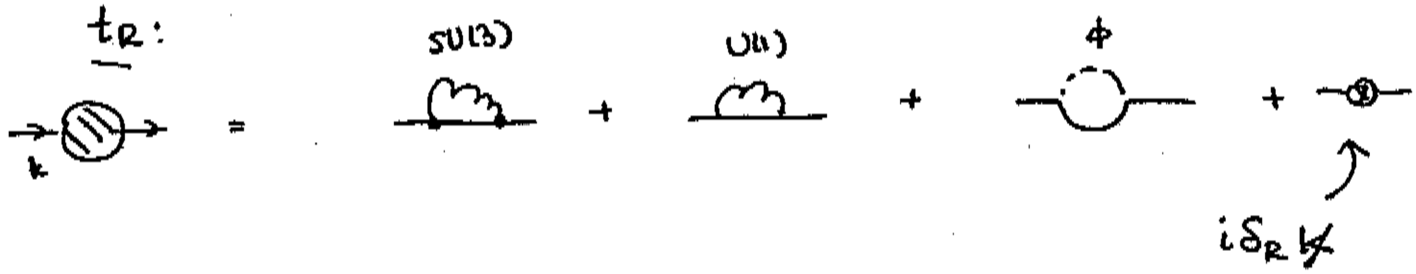
$$m_t(m_t) = 167.\text{ GeV} = 167\text{ GeV}$$

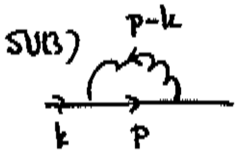
so the more intrinsic quark mass ratios are, eg.

$$\frac{m_b}{m_t} = \frac{1}{58.} \qquad \frac{m_c}{m_b} = \frac{1}{4.3}$$

4.) To compute the $\beta$-function, work in an $SU(3)\times SU(2)\times U(1)$ gauge thy with all symmetries restored. We need the $t_L$, $t_R$, $\phi$ self-energies and the $t_L t_R \phi$ vertex corrections. Use Feynman-'t Hooft gauge.

$t_R$:




$$= (ig_s)^2 t^a t^a \int \frac{d^dp}{(2\pi)^d} \frac{-i}{(p-k)^2}\, \gamma_\mu \frac{i\not p}{p^2} \gamma^\mu$$

$$= -g_s^2 \frac{4}{3} \int_0^1 dx \int \frac{d^d P}{(2\pi)^4} \frac{1}{[P^2-\Delta]^2} (2-d) \not p$$

$$P = p - xk \qquad p = P + xk$$

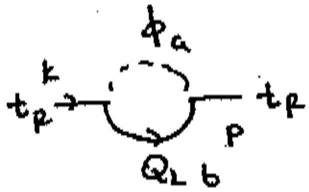
$$= -g_s^2 \frac{4}{3} \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) (-2-\epsilon) \times \not k$$

$$= -g_s^2 \frac{4}{3} (-2) \frac{1}{2} \not k \frac{i}{(4\pi)^2} \cdot \frac{2}{\epsilon} = i \frac{g_s^2}{(4\pi)} \cdot \frac{4}{3} \cdot \frac{2}{\epsilon} \not k$$

similarly


$$= i \frac{g'^2}{(4\pi)} \cdot Y^2 \cdot \frac{2}{\epsilon} \not{p}$$

$$= i \frac{g'^2}{4\pi} \cdot \left(\frac{2}{3}\right)^2 \frac{2}{\epsilon} \not{p}$$




$$= (-i \lambda_t \epsilon^{ab})(-i \lambda_t \epsilon^{ab}) \int \frac{d^d p}{(2\pi)^d} \frac{i \not{p}}{p^2} \frac{i}{(k-p)^2}$$

$$= \lambda_t^2 \cdot 2 \cdot \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{[\ell^2-\Delta]^2} \times \not{p}$$

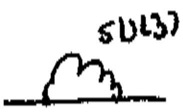
$$= \lambda_t^2 \cdot 2 \cdot \tfrac{1}{2} \not{p} \left( \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \right)$$

$$= i \frac{\lambda_t^2}{(4\pi)^2} \not{p} \frac{2}{\epsilon}$$

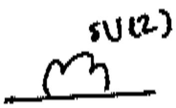
so for $t_R$

$$\delta_R = \left[ - \frac{g_s^2}{(4\pi)^2} \cdot \frac{4}{3} - \frac{g'^2}{(4\pi)^2} \cdot \frac{4}{9} - \frac{\lambda_t^2}{(4\pi)^2} \right] \frac{2}{\epsilon}$$

$\underline{t_L}$:

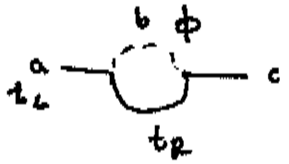



$$= i \frac{g_s^2}{(4\pi)^2} \cdot \frac{4}{3} \frac{2}{\epsilon} \not{p}$$




$$= i \frac{g^2}{(4\pi)^2} C_2(r) \frac{2}{\epsilon} \not{p} = i \frac{g^2}{(4\pi)^2} \frac{3}{4} \frac{2}{\epsilon} \not{p}$$


$$= i \frac{g'^2}{(4\pi)^2} Y^2 \frac{2}{\epsilon} \not{p} = i \frac{g'^2}{(4\pi)^2} \left(\frac{1}{6}\right)^2 \frac{2}{\epsilon} \not{p}$$



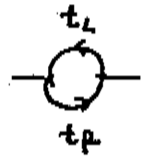

$$= (-i\lambda_t\epsilon^{ab})(-i\lambda_t\epsilon^{cb})\int\frac{d^dp}{(2\pi)^d}\,\frac{i\not p}{p^2}\,\frac{i}{(k-p)^2}$$

$$= i\,\frac{\lambda_t^2}{(4\pi)^2}\,\underbrace{\left(\frac{\delta^{ac}}{2}\right)}\,\not k\,\frac{2}{\epsilon}$$

only charge from $t_R$

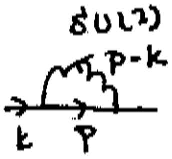
$$\delta_L = \left[-\frac{g_s^2}{(4\pi)^2}\cdot\frac{4}{3} - \frac{g^2}{(4\pi)^2}\frac{3}{4} - \frac{g'^2}{(4\pi)^2}\cdot\frac{1}{36} - \frac{\lambda_t^2}{(4\pi)^2}\cdot\frac{1}{2}\right]\frac{2}{\epsilon}$$

$\phi$:

 =  +  +  + 



 has no term $\propto k^2$




$$= (+ig)^2\,t^at^a\int\frac{d^dp}{(2\pi)^d}\,\frac{i}{p^2}\,(p+k)^\mu(p+k)_\mu\,\frac{-i}{(p-k)^2}$$

$$= -g^2C_2(r)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\,\frac{(p+k)^2}{[\ell^2+x(1-x)k^2]^2} \qquad p+k = \ell+(1+x)k$$

$$= -g^2C_2(r)\int_0^1 dx\,\frac{i}{(4\pi)^{d/2}}\left\{-\frac{\Gamma(1-d/2)}{[-x(1-x)k^2]^{1-d/2}}\cdot\frac{d}{2}\right.$$

$$\left.+\frac{\Gamma(2-d/2)}{[-x(1-x)k^2]^{2-d/2}}\,(1+x)^2k^2\right\}$$

$$= -g^2C_2(r)\int_0^1 dx\,\frac{i}{(4\pi)^{d/2}}\frac{1}{[-x(1-x)k^2]^{2-d/2}}\,\Gamma(2-d/2)$$

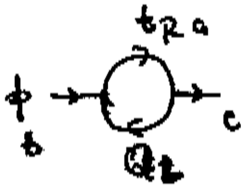
$$\cdot\left\{-\frac{1}{1-d/2}\,\frac{d}{2}\,(-x(1-x)k^2) + (1+x)^2k^2\right\}$$

$$= -i\,\frac{g^2 C_2(r)}{(4\pi)^2}\,\frac{2}{\epsilon}\int_0^1 dx\,\left[(-2x+2x^2)k^2 + (1+2x+x^2)k^2\right]$$

pole at $\epsilon = 0$

$$= -i\,\frac{g^2 C_2(r)}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot 2\cdot k^2$$

$$= -i\,\frac{2g^2 C_2(r)}{(4\pi)^2}\,\frac{2}{\epsilon}\,k^2 \qquad C_2(r) = \begin{cases} SU(2) \quad 3/4 \\ Y^2 = \frac{1}{4} \quad U(1) \end{cases}$$




$$= (-i\lambda_t\,\varepsilon^{ac})(-i\lambda_t\,\varepsilon^{ab})\,\underset{\text{fermions}}{(-1)}\,\underbrace{\mathrm{tr}[1]}_{\text{color}}$$

$$\cdot\int\frac{d^dp}{(2\pi)^d}\,\mathrm{tr}\left[\left(\frac{1+\gamma^5}{2}\right)\frac{i\not p}{p^2}\left(\frac{1-\gamma^5}{2}\right)\frac{i(\not p+\not k)}{(p+k)^2}\right]$$

$$= -\lambda_t^2\,\delta^{bc}\cdot 3\int_0^1 dx\int\frac{d^d P}{(2\pi)^d}\,\frac{1}{[P^2+x(1-x)k^2]^2}\;2\,p\cdot(p+k)$$

$$P = p+xk \qquad p = P - xk \qquad (p+k) = P + (1-x)k$$

$$= -\lambda_t^2\,\delta^{bc}\cdot 3\cdot\int_0^1 dx\,\frac{i}{(4\pi)^{d/2}}\cdot 2\cdot\left\{-\frac{\Gamma(1-d/2)}{[-x(1-x)k^2]^{1-d/2}}\cdot\frac{d}{2}\right.$$

$$\left. + \frac{\Gamma(2-d/2)}{[-x(1-x)k^2]^{2-d/2}}\,[-x(1-x)k^2]\right\}$$

$$= -3\lambda_t^2\,\delta^{bc}\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\cdot 2\int_0^1 dx\,\frac{1}{[-x(1-x)k^2]^{2-d/2}}$$

$$\cdot\left\{-\frac{1}{1-d/2}\cdot\frac{d}{2}\cdot(-x(1-x)k^2) - x(1-x)k^2\right\}$$

$$\text{pole at } d=4 \quad = \quad -i\, 6 \lambda_t^2\, \delta^{bc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \int_0^1 dx \left[ -2x(1-x)\, k^2 - x(1-x)\, k^2 \right]$$

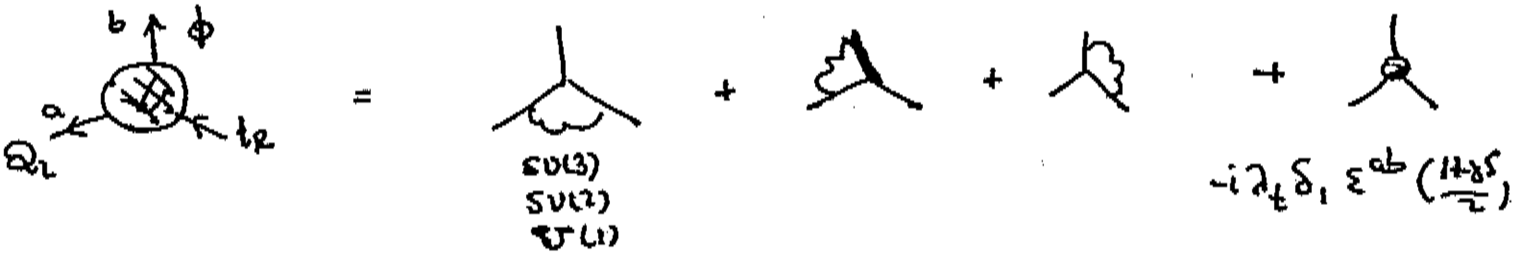
$$= \quad +i\, 6 \lambda_t^2\, \delta^{bc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \cdot 3 \cdot \frac{1}{6} \cdot k^2$$

$$= \quad +3 \lambda_t^2\, \delta^{bc} \frac{1}{(4\pi)^2} \frac{2}{\epsilon}\, k^2$$

so

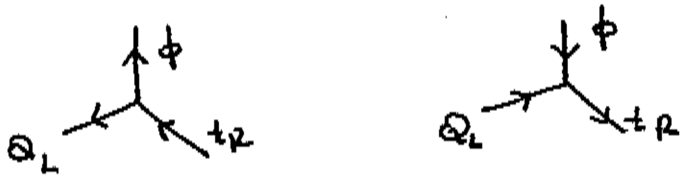
$$\delta_\phi = \frac{1}{(4\pi)^2} \left[ 2 \cdot \left( \frac{3}{4} g^2 + \frac{1}{4} g'^2 \right) - 3 \lambda_t^2 \right] \frac{2}{\epsilon}$$

finally, we need the radiative corrections to the vertex:

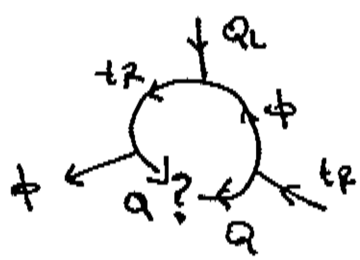



note ~~that~~ ~~there~~ is no vertex correction $\propto \lambda_t^3$

the vertices are

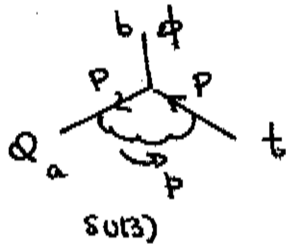



so




is inconsistent.

We can evaluate the gauge diagrams w. zero external momentum.




$$= (-i\lambda_t)\, \varepsilon^{ab} (t^A t^A)(ig_s)^2 \int \frac{d^dp}{(2\pi)^d}\, \gamma^\mu \frac{i\not{p}}{p^2} \frac{1+\gamma^5}{2} \frac{i\not{p}}{p^2} \gamma_\mu \frac{-i}{p^2}$$

$$= -i\lambda_t\, \varepsilon^{ab}\, \frac{4}{3} (-ig_s^2) \int \frac{d^dp}{(2\pi)^d}\, \gamma^\mu \frac{p^2}{(p^2)^3} \gamma_\mu \left(\frac{1+\gamma^5}{2}\right)$$

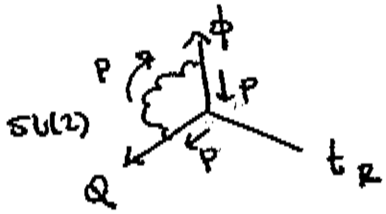
$$= -i\lambda_t\, \varepsilon^{ab}\, \frac{4}{3} (-ig_s^2) \cdot d \cdot \frac{i}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) \left(\frac{1+\gamma^5}{2}\right)$$

$$= (-i\lambda_t\, \varepsilon^{ab})\, \frac{4}{3} \frac{4g_s^2}{(4\pi)^2} \frac{2}{\epsilon} \left(\frac{1+\gamma^5}{2}\right)$$

for SU(2), the diagram is <u>zero</u>

for U(1) $\quad (t^A t^A) \to \frac{1}{6} \cdot \frac{2}{3} = \frac{1}{9}$


$$= -i\lambda_t\, \varepsilon^{ab} \left(\frac{1+\gamma^5}{2}\right) \cdot \frac{1}{9} \frac{4g'^2}{(4\pi)^2} \frac{2}{\epsilon}$$




$$= (-i\lambda_t)(ig)^2 (\tau^i)_{aa'} (\tau^i)_{bb'}\, \varepsilon^{a'b'}$$

$$\int \frac{d^dp}{(2\pi)^d}\, \gamma_\mu \frac{i\not{p}}{p^2} \left(\frac{1+\gamma^5}{2}\right) \cdot \frac{i}{p^2} \frac{-i}{p^2} \cdot (-p^\mu)$$

$$= (-i\lambda_t)\; ig^2 \int \frac{d^dp}{(2\pi)^d} \frac{\not{p}\not{p}}{(p^2)^3} (\tau^i)_{aa'}\, \varepsilon^{a'b'} [(\tau^i)^T]_{b'b} \left(\frac{1+\gamma^5}{2}\right)$$

$$= (-i\lambda_t)\; ig^2 \left(\frac{i}{(4\pi)^2} \frac{2}{\epsilon}\right) [\tau^i(-\tau^i)\varepsilon]^{ab} \left(\frac{1+\gamma^5}{2}\right)$$

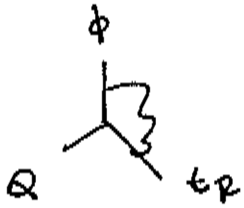
$$= (-i\lambda_t) \left(-\frac{g^2}{(4\pi)^2} \frac{2}{\epsilon}\right) \cdot \left(-\frac{3}{4}\right) \varepsilon^{ab} \left(\frac{1+\gamma^5}{2}\right)$$

$$= (-i\lambda_t)\, \varepsilon^{ab} \left(\frac{1+\gamma^5}{2}\right) \left(\frac{3}{4} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon}\right)$$

for SU(3), this diagram is zero

for U(1) $\quad (\tau^i)_{aa'} (\tau^i)_{bb'} \, \varepsilon^{a'b'} \longrightarrow \varepsilon^{ab} \cdot \frac{1}{6} \cdot \frac{1}{2}$

$$= \left[ -i\lambda_t \, \varepsilon^{ab} \left( \frac{1+\gamma^5}{2} \right) \right] \cdot \left( -\frac{1}{12} \frac{g'^2}{(4\pi)^2} \frac{2}{\epsilon} \right)$$




is nonzero for U(1) only:

$$= \left[ -i\lambda_t \, \varepsilon^{ab} \left( \frac{1+\gamma^5}{2} \right) \right] (ig')^2 \int \frac{d^d p}{(2\pi)^d} \left( \frac{i\not{p}}{p^2} \gamma_\mu \right) \frac{-i}{p^2} \frac{i}{p^2} \, p^\mu$$

$$\cdot \frac{1}{2} \cdot \frac{2}{3}$$

$$= \left[ -i\lambda_t \, \varepsilon^{ab} \left( \frac{1+\gamma^5}{2} \right) \right] (-ig'^2) \frac{i}{(4\pi)^2} \frac{2}{\epsilon} \cdot \frac{1}{3}$$

$$= \left( -i\lambda_t \, \varepsilon^{ab} \left( \frac{1+\gamma^5}{2} \right) \right) \frac{1}{3} \frac{g'^2}{(4\pi)^2} \frac{2}{\epsilon}$$

in all

$$\delta_1 = -\frac{16}{3} \frac{g_s^2}{(4\pi)^2} \frac{2}{\epsilon} - \frac{3}{4} \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon}$$

$$- \Big[ \underbrace{\frac{4}{9} - \frac{1}{12} + \frac{1}{3}}_{\frac{25}{36}} \Big] \frac{g'^2}{(4\pi)^2} \frac{2}{\epsilon}$$

Now combine all of these results:

$$\beta = M\frac{\partial}{\partial M}\left[-\lambda_t\delta_1 + \lambda_t\cdot\tfrac{1}{2}(\delta_R+\delta_L+\delta_\phi)\right]$$

$$= -\lambda_t\, M\frac{\partial}{\partial M}\Big\{\left[\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right]$$

$$\cdot\Big[\left(-\tfrac{16}{3}g_s^2 - \tfrac{3}{4}g^2 - \tfrac{25}{36}g'^2\right)$$

$$-\tfrac{1}{2}\left(-\tfrac{4}{3}g_s^2 - \tfrac{4}{9}g'^2 - \lambda_t^2\right)$$

$$-\tfrac{1}{2}\left(-\tfrac{4}{3}g_s^2 - \tfrac{3}{4}g^2 - \tfrac{1}{36}g'^2 - \tfrac{1}{2}\lambda_t^2\right)$$

$$-\tfrac{1}{2}\left(\tfrac{3}{2}g^2 + \tfrac{1}{2}g'^2 - 3\lambda_t^2\right)\Big]\Big\}$$

$$2/\epsilon \to \log\Lambda^2/M^2 \quad\text{so}\quad M\frac{\partial}{\partial M}(2/\epsilon) = -2$$

$$\beta = 2\lambda_t\cdot\frac{1}{(4\pi)^2}\Big\{g_s^2\left(-\tfrac{16}{3}+\tfrac{4}{3}\right)$$

$$+\,g^2\left(-\tfrac{3}{4}+\tfrac{3}{8}-\tfrac{3}{4}\right)$$

$$+\,g'^2\left(-\tfrac{25}{36}+\tfrac{2}{9}+\tfrac{1}{72}-\tfrac{1}{4}\right)$$

$$+\,\lambda_t^2\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{3}{2}\right)\Big\}$$

$$= \frac{2\lambda_t}{(4\pi)^2}\Big\{-4g_s^2 - \tfrac{9}{8}g^2 - \tfrac{51}{72}g'^2 + \tfrac{9}{4}\lambda_t^2\Big\}$$

$$\Downarrow$$

$$\tfrac{17}{24}$$

$$\beta = \frac{\lambda_t}{(4\pi)^2}\left\{ \frac{9}{2}\lambda_t^2 - 8g_s^2 - \frac{9}{4}g^2 - \frac{17}{12}g'^2 \right\}$$

as requested.

If we consider only the $\lambda_t$ and $g_s$ contributions, the $\lambda_t^3$ term makes $\lambda_t$ <u>decrease</u> toward the infrared; the $\lambda_t g_s^2$ term makes $\lambda_t$ increase toward the infrared.

Balance is found at

$$\frac{9}{2}\lambda_t^2 \cong 8g_s^2$$

$$\frac{9}{8\pi}\lambda_t^2 \cong 8\alpha_s$$

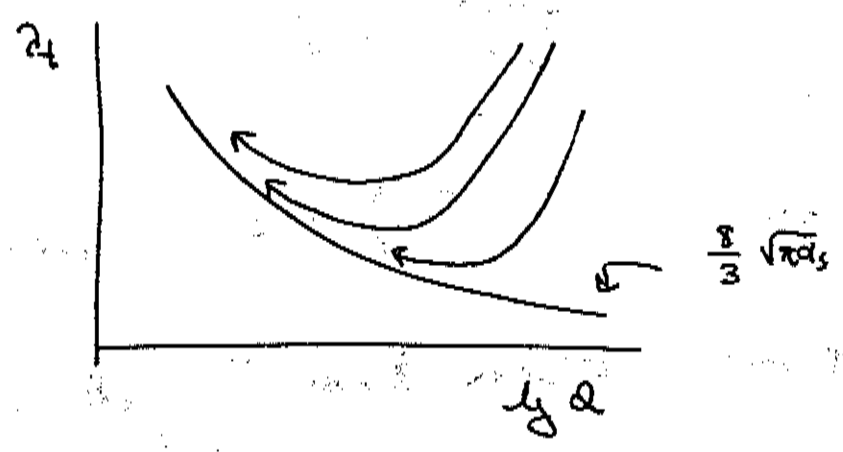
$$\lambda_t^2 \approx \frac{64\pi}{9}\alpha_s$$

Of course, $\alpha_s$ runs, but if it runs sufficiently slowly

$$\lambda_t(m_t) \approx \frac{8}{3}\sqrt{\pi\alpha_s(m_t)} \approx 1.6 \qquad \alpha_s(m_t) = 0.11$$

$$m_t \approx \frac{\lambda_t}{\sqrt{2}}v \approx 270\text{ GeV} \qquad v = 246\text{ GeV}$$

~~Integrating~~ the differential equation for $\alpha_s$, $\lambda_t$, one finds that, for $\lambda_t$ large at $Q \gg 1000$ GeV, this asymptote is approximately reached




5.) Using $m_p = (\text{const}) \cdot \Lambda_3$

treating u, d, s as light quarks, c b t as heavy quarks at the proton scale

$$m_p \propto (m_t\, m_b\, m_c)^{2/27}$$

$$\frac{\partial m_p}{\partial m_t} = \frac{2}{27}\,\frac{m_p}{m_t}$$

since all three of $m_t, m_b, m_c$ are $\propto v$

$$\frac{\partial m_p}{\partial v} = 3 \cdot \frac{2}{27} \cdot \frac{m_p}{v}$$

$$= \frac{2}{9} \cdot \frac{0.938}{246.}$$

$$\lambda_{hpp} = \sqrt{2} \cdot \frac{2}{9} \frac{m_p}{v} = 1.2 \times 10^{-3}$$

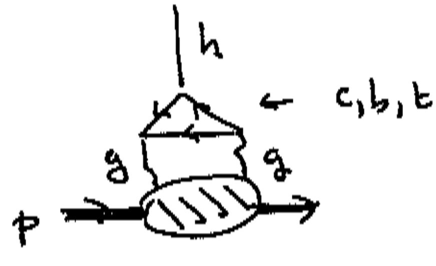
Compare this to the direct contribution from the valence quarks u and d

$$\Delta m_p = 2m_u + m_d$$

$$\frac{\partial}{\partial v} \Delta m_p = \frac{2m_u + m_d}{v} = \frac{(2 \cdot 3 + 5) \cdot 10^{-3}}{246}$$

$$\Delta \lambda_{hpp} = \sqrt{2} \cdot (\text{above}) = 6 \times 10^{-5}$$

the large contribution to $\lambda_{hpp}$ can be thought of as coming from the diagram




this diagram is constant as $m_q \to \infty$ !