Physics 331 – Problem Set # 9

(due Wednesday, March 11)

1. An illustrative example of spontaneous breaking of a gauge symmetry is given by a theory called *topcolor* in which, at very short distances, the (t, b) quarks transform under a different $SU(3)$ color group from the lighter quarks.

   (a) The gauge group of the topcolor theory is $SU(3)_1 \times SU(3)_2$. The theory has two sets of 8 gauge bosons and two independent coupling constants g_1, g_2. The light quarks (u, d, s, c) transform only under $SU(3)_1$ according to

   $$q_u \to (1 + i\alpha_1^a t^a) q_u \ , \tag{1}$$

   where t^a is a 3×3 representation matrix for $SU(3)$. The b and t quarks transform similarly under $SU(3)_2$,

   $$q_t \to (1 + i\alpha_2^a t^a) q_t \ , \tag{2}$$

   The model also contains a complex-valued scalar field Φ which is a (3×3) matrix and transforms as

   $$\Phi \to \Phi + (i\alpha_1^a t^a)\ \Phi + \Phi(-i\alpha_2 t^a) \ . \tag{3}$$

   Show that the covariant derivatives of the theory are

   $$\begin{aligned} D_\mu q_u &= (\partial_\mu - i g_1 A_{\mu 1}^a t^a) q_u \ , \\ D_\mu q_t &= (\partial_\mu - i g_2 A_{\mu 2}^a t^a) q_t \ , \\ D_\mu \Phi &= \partial_\mu \Phi - i(g_1 A_{\mu 1}^a t^a)\Phi + \Phi(+i g_2 A_{\mu 2}^a t^a) \ . \end{aligned} \tag{4}$$

   The Lagrangian of the topcolor theory is (ignoring the light quark masses)

   $$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(F_1^{\mu\nu a})^2 - \frac{1}{4}(F_2^{\mu\nu a})^2 \\ &\quad + \sum_{f=u,d,s,c} \overline{q}_f i\gamma \cdot D q_f + \sum_{f'=b,t} \overline{q}_{f'}(i\gamma \cdot D - m_{f'}) q_{f'} \\ &\quad + \mathrm{tr}[(D_\mu \Phi)^\dagger D^\mu \Phi] - V(\Phi) \ . \end{aligned} \tag{5}$$

   (b) Assume that the minimum of the potential $V(\Phi)$ is at the nonzero value

   $$\langle \Phi \rangle = V \cdot \mathbf{1} \ , \tag{6}$$

   where $\mathbf{1}$ is the 3×3 unit matrix and V is a constant with the dimensions of mass. Find the mass terms for $A_{\mu 1}^a$ and $A_{\mu 2}^a$. Show that one linear combination of $A_{\mu 1}^a$ and $A_{\mu 2}^a$ remains massless. Why must this be so?

(c) Construct the normalized mass eigenstate fields. Call the new massless and massive vector fields, respectively, A^a_μ and $\mathbf{A}^a_\mu$. Show that the massive field has the mass

$$m^2 = (g_1^2 + g_2^2)V^2 \ . \tag{7}$$

(d) Rewrite the covariant deratives on the quark fields in terms of the mass eigenstate vector fields. Show that all quarks now couple to the field A^a_μ with the same coupling constant g, given by

$$\frac{1}{g^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2} \ . \tag{8}$$

So we find an $SU(3)$ gauge theory just like QCD, with a coupling g smaller than either g_1 or g_2. This property that coupling constants combine like resistors in parallel is often seen in models with spontaneous gauge symmetry breaking.

(e) The heavy boson $\mathbf{A}^a_\mu$ will appear as a resonance in proton-proton collisions. Compute the decay rate for this resonance into (1) a pair of gluons, (2) a pair of light quarks (e.g., $u\overline{u}$), (3) a pair of heavy quarks (e.g. $t\overline{t}$).

2. Consider further the Georgi-Glashow model in which a Higgs field ϕ^a in the **3** representation of $SU(2)$ breaks the symmetry of an $SU(2)$ theory to $U(1)$. In this theory, two of the original gauge bosons obtain mass. Then, as explained in class, the model can be viewed as a unified model of weak and electromagnetic interactions. However, this is not the model chosen by nature.

(a) Consider first the solution to the theory with $\langle \phi^a \rangle = v\delta^{a3}$. Using

$$D_\mu \phi^a = \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c \tag{9}$$

show that the bosons A^1_μ, A^2_μ obtain the mass $m_W = gv$ and the boson A^3_μ obtains zero mass.

(b) Couple a fermion in the spinor (**2**) representation of $SU(2)$ to this gauge field. Write the covariant derivative on this fermion in terms of the boson mass eigenstates. Identifying A^3_μ with the photon, find the electric charges of the two fermions. Call these $\pm e$, and find e in terms of g.

(c) Now consider a solution to the theory in which ϕ^a takes the form

$$\phi^a(r) = vf(r)\hat{r}^a \ , \tag{10}$$

where r is the radial distance in 3-d space, $\hat{r}^i$ is a unit vector in the radial direction, and $f(r)$ is a function to be determined. Show that, if $f(r) \to 1$ as $r \to \infty$, then, for large r in any direction, $\phi^a(r)$ tends to a vacuum configuration for ϕ^a.

(d) A compatible ansatz for A^a_μ is

$$A^b_i = -\frac{1}{gr}\epsilon^{ibk}\hat{r}^k h(r) \ . \tag{11}$$

Here i is a *lowered* spatial index. Compute $D_i\phi^a$. Show that, as $r \to \infty$, each term of the covariant derivative decreases like $1/r$ but these leading terms cancel if $h(r) \to 1$ as $r \to \infty$. Then the covariant derivative decreases like $1/r^2$. Since the energy of the scalar field contains a term

$$\int d^3x \ \frac{1}{2}(D_i\phi^a)^2 \ , \tag{12}$$

this is necessary to have finite energy.

(e) Compute $F_{ij} = \partial_i A^a_j - \partial_j A^a_i + g\epsilon^{abc}A^b_i A^c_j$ in terms of the unknown function $h(r)$. You will find a rather long expression. Compute the corresponding magnetic field

$$B^{ka} = -\frac{1}{2}\epsilon^{ijk}F^a_{ij} \tag{13}$$

You will find a more compact expression.

(f) Assuming that $h'(r) \to 0$ rapidly as $r \to \infty$, show that

$$B^{ka} \to \hat{r}^a \frac{\hat{r}^k}{gr^2} \tag{14}$$

(In the full solution, $h(r) \sim 1 - \mathcal{O}(e^{-m_w r})$ as $r \to \infty$.)

(g) Far away from $r = 0$, the physical E and B fields of the effective QED theory are those in the direction in the gauge space parallel to ϕ^a or $\hat{r}^a$. Then the physical B field at large distances is

$$\vec{B} = \frac{\hat{r}}{gr^2} \tag{15}$$

This is the field of a magnetic monopole. Compute the magnetic charge in terms of e in part (b). Look up “Dirac quantization of the magnetic monopole” and compare the results.

This construction was discovered by Gerard ‘t Hooft and Alexander Polyakov.