Physics 331 – Problem Set # 8

(due Wednesday, March 4)

1. In class, we derived the Altarelli-Parisi splitting function for $q \to g$,

$$P_{g\leftarrow q}(z) = \frac{4}{3}\frac{1+(1-z)^2}{z} \ . \tag{1}$$

You can also find the derivation in Peskin and Schroeder, Section 17.5, and in Schwartz, Section 32.2. Using explicit helicity spinors and polarization vectors with the QCD 3-point vertices and changing only what needs to be changed, derive the splitting functions

$$\begin{aligned} P_{q\leftarrow g}(z) &= \frac{1}{2}[z^2 + (1-z)^2] \\ P_{g\leftarrow g}(z) &= 3\ [\frac{z^4 + (1-z)^4 + 1}{z(1-z)}] \ . \end{aligned} \tag{2}$$

2. Compute the exact differential cross section for $e^+e^- \to q\overline{q}g$ at the tree level in QCD. This is a somewhat long calculation, but it is made easier with a number of tricks. Let the quark, antiquark, and gluon momenta be p_1, p_2, and p_3, respectively. Let $Q^\mu = p_1^\mu + p_2^\mu + p_3^\mu$, so that $Q = \sqrt{Q^2}$ is the CM energy. Let the initial lepton moment be k^μ, $\overline{k}^\mu$. Ignore the masses of the quarks and the electrons.

   (a) Draw the 2 Feynman diagrams contributing to this cross section. Show that they satisfy the Ward identity, that is, replacing the gluon polarization vector by p_3^μ gives zero. This allows us to do the gluon polarization sum by

   $$\sum_\epsilon \epsilon^\mu(p_3)\epsilon^{\nu *}(p_3) = -g^{\mu\nu} \tag{3}$$

   (b) Let

   $$x_i = \frac{2p_i \cdot Q}{Q^2} \tag{4}$$

   for $i = 1, 2, 3$. Show that $x_1 + x_2 + x_3 = 2$. Show that x_i has the range $[0, 1]$ and corresponds to the ratio of the CM energy of the particle i to its maximum possible value $Q/2$. Describe the configurations in which $x_1 = 1$, $x_2 = 1$, $x_3 = 1$.

   (c) The most general configuration of the 3 particles is that they lie in a plane in the CM system. Show that the angle between the quark and antiquark θ_{12} is given by

   $$\frac{1}{2}\,x_1x_2\,(1-\cos\theta_{12}) = (1-x_3)\ , \tag{5}$$

   and similarly for the other angles in this plane.

(d) Write the expression for 3-body massless phase space. Integrate over all angles except θ_{12}. It is possible also to integrate over θ_{12} using the energy conservation δ function. In this way, show that

$$\int d\Pi_3 = \frac{Q^2}{128\pi^3} \int dx_1\, dx_2 \ . \tag{6}$$

(e) Using the insight from part (b), show that the integration region is the region of the square $0 \le x_1, x_2 \le 1$ above the line $x_1 + x_2 = 1$.

(f) The expression for the initial lepton spinors averaged over polarizations is

$$\frac{1}{4}\mathrm{tr}[\overline{v}(\overline{k})\gamma^\mu u(k)\ \overline{u}(k)\gamma^\nu v(\overline{k})] \tag{7}$$

Evaluate the trace, write this out as a 4×4 matrix, average it over orientations of the initial electron-positron beam direction, and show that the result is

$$-\frac{Q^2}{3}[g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}] \ . \tag{8}$$

(g) Check your result in part (e) by rederiving the total cross section with unpolarized beams

$$\sigma(e^+e^- \to q\overline{q}) \equiv \sigma_0 = \frac{4\pi\alpha^2}{3Q^2} \cdot 3\mathcal{Q}^3 \ , \tag{9}$$

where 3 is the number of colors and $\mathcal{Q}$ is the quark charge.

(h) Now put all of the pieces together and derive the formula

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\overline{q}g) = \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \ . \tag{10}$$

(i) Take the limit of (10) as $x_2 \to 1$ (in which the antiquark recoils against an almost collinear qg system) and show that this agrees with the expectation from Altarelli-Parisi splitting.