Physics 331 – Problem Set # 7

(due Wednesday, February 26)

1. The Gross-Neveu model is a model of fermions with a discrete chiral symmetry in two spacetime dimensions,

$$\mathcal{L} = \overline{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 (\overline{\psi}_i \psi_i)^2 \tag{1}$$

The kinetic term involves matrices γ^μ, $\mu = 0, 1$, satisfying the 2-dimensional Dirac algebra. These matrices can be 2×2. A convenient choice is

$$\gamma^0 = \sigma^2 \qquad \gamma^1 = i\sigma^1 \tag{2}$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0 \gamma^1 = \sigma^3 \tag{3}$$

This anticommutes with γ^0, γ^1. We will analyze this theory to 1-loop order (order g^2).

   (a) Show that this theory is invariant under the symmetry

$$\psi_i \to \gamma^5 \psi_i \tag{4}$$

   and that this symmetry forbids a fermion mass.

   (b) Show that g is dimensionless, so that the theory is renormalizable in 2 dimensions.

   (c) Show that the functional integral for this theory can be represented in the following form:

$$\int D\overline{\psi} D\psi e^{i\int \mathcal{L}} = \int D\overline{\psi} D\psi D\sigma \ \exp[i \int d^2x \left\{ \overline{\psi}_i i \not{\partial} \psi_i - \sigma \overline{\psi}_i \psi_i - \frac{1}{2g^2} \sigma^2 \right\}] \ , \tag{5}$$

   where $\sigma(x)$ is a new scalar field introduced as a Lagrange mulitiplier, with no kinetic energy term.

   (d) Integrate over the ψ and $\overline{\psi}$ fields and evaluate the resulting functional determinant. The easiest way to do this is to Fourier analyze to obtain an integral d^2p over the log of the determinant of a 2×2 matrix. Evaluate the determinant, then calculate the integral over momenta using dimensional regularization. The formula (Euclidean space)

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + m^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - d/2}} \tag{6}$$

   might be useful. Replace $1/\epsilon \to \log \Lambda^2/\mu^2$, where $\epsilon = 1 - d/2$, to get a clearer physical picture of the result.

(e) The evaluation of the determinant gives a potential for the field σ. Minimize this potential, and show that $\sigma(x)$ has a nonzero value at this minimum. Assume that the value of g in the Lagrangian (or, g at the cutoff scale Λ) is very small How does $\langle\sigma\rangle$ depend on g?

(f) Compare this result to the hierarchy formula in asymptotically free Yang-Mills theory.

2. At the beginning of the term, we derived the representation of the propagator of the Schrödinger equation

$$\langle x_f| e^{-HT} |x_i\rangle = \int \mathcal{D}x(t) \exp\Big[-\int dt(\frac{1}{2}(\dot{x})^2 + V(x))\Big] \ , \tag{7}$$

where $H = -\frac{1}{2}\nabla^2 + V$, and the integral is taken over paths $\{x(t)\}$ that go from x_i to x_f in time T. Let's use this to get some representations of propagators in quantum field theory in Euclidean space.

(a) Write the analogue of eq. (7) for paths in 4-dimensional Euclidean space with $V(\vec{x}) = \frac{1}{2}m^2$. Using the identity

$$\frac{1}{X} = \int_0^\infty dT\ e^{-XT} \tag{8}$$

show that this gives a representation of the propagator of the Klein-Gordon equation in Euclidean space

$$\langle x_f| \frac{1}{k^2+m^2} |x_i\rangle \tag{9}$$

Note that the particle paths integrated over can bend back and forth in Euclidean time.

(b) Next, consider generalizing this to the propagator of the Klein-Gordon operator coupled to a $U(1)$ gauge field

$$-(\partial_\mu - igA_\mu)^2 + m^2 \tag{10}$$

By repeating the derivation of eq. (7), show that this propagator is given by the same formula with a Wilson line $\exp[+ig\int d\vec{x}\cdot\vec{A}(x)]$ along the path.

(c) Generalize the result of (b) to a Klein-Gordon field coupled to a non-Abelian gauge field. Show that the representation contains a path-ordered Wilson line running along the path of the particle.

(d) To get a similar representation of the Dirac propagator, we can square the Dirac equation as we did in the background field method to produce a Klein-Gordon operator with an extra spin term. Show that, with Euclidean γ matrices satisfying

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} \ , \tag{11}$$

one can represent

$$-(\not{D}+m)(\not{D}-m) = -D^2 + m^2 + 2i(\frac{1}{2}S_{mn}F^{mna}t^a) \ , \tag{12}$$

where S_{mn} is a spin operator for the fermion. The factor 2 is just $g = 2$ for the Dirac field.

(e) Write the functional representation of the propagator of the operator (12). Be explicit about the ordering of t^a matrices along the path.