Physics 331 – Problem Set # 6

(due Wednesday, February 19)

1. The lowest-order scattering amplitudes in Yang-Mills theory have interesting regularities that are best exposed by computing amplitudes for quarks and gluons of definite helicity.

   In this problem, I will specialize to QCD, a theory with gauge group $SU(3)$, quarks q such as u and d in the $\mathbf{3}$ representation, and antiquarks in the $\overline{\mathbf{3}}$ representation. Recall that, for $SU(3)$, $C_2(G) = C(G) = 3$ and $C_2(\mathbf{3}) = \frac{4}{3}$. You will also encounter

$$\mathrm{tr}[t^a t^b t^a t^b] = 3C_2(\mathbf{3})(C_2(\mathbf{3}) - \frac{1}{2}C_2(G)) = -\frac{2}{3} \tag{1}$$

   In this problem, use the Feynman rules derived in class with explicit spinors and polarization vectors to compute the amplitudes. Set the quark masses equal to zero. For massless left-handed quarks and right-handed antiquarks moving in the $\hat{3}$ diretion, the spinors are

$$u_- = v_+ = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{2}$$

   For massless left-handed quarks and right-handed antiquarks moving in the direction $\cos\theta\hat{3} + \sin\theta\hat{1}$, the spinors are

$$u_- = v_+ = \sqrt{2E}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta \\ 0 \\ 0 \end{pmatrix} . \tag{3}$$

   For massless gluons moving in the $\hat{3}$ direction, the transverse polarization vectors are

$$\epsilon_+ = \frac{1}{\sqrt{2}}(0, 1, i, 0) \qquad \epsilon_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \tag{4}$$

   For massless gluons moving in the $\cos\theta\hat{3} + \sin\theta\hat{1}$, the transverse polarization vectors are

$$\epsilon_+ = \frac{1}{\sqrt{2}}(0, \cos\theta, i, -\sin\theta) \qquad \epsilon_- = \frac{1}{\sqrt{2}}(0, \cos\theta, -i, -\sin\theta) \tag{5}$$

   To view helicity symmetrically between initial and final states, I will, in the following, count a final-state left-handed $(-)$ particle as an incoming right-handed antiparticle $(+)$ and a final-state right-handed $(+)$ particle as an incoming left-handed $(-)$ antiparticle. Then, for example, $q_L\overline{q}_R \to q_L\overline{q}_R$ will be denoted as a $(-++-)$ amplitude.

   (a) Prove the identity (1).

(b) Show that, for reactions $qq \to qq$, $q\overline{q} \to q\overline{q}$, chirality conservation implies that only processes with two + helicities and two − helicities have nonzero scattering amplitudes.

(c) Compute the nonzero scattering amplitudes contributing to $ud \to ud$. Square, sum over final helicities and colors and average over initial helicities and colors. Show that, in the center of mass frame,

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha_s^2}{9s}\left(\frac{s^2+u^2}{t^2}\right) . \tag{6}$$

(d) Compute the nonzero scattering amplitudes contributing to $uu \to uu$. Don't forget the minus sign from fermion exchange. Square, sum over final helicities and colors and average over initial helicities and colors. Show that, in the center of mass frame,

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha_s^2}{9s}\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - \frac{2}{3}\frac{s^2}{tu}\right) . \tag{7}$$

(e) For the process $q\overline{q} \to gg$, chirality conservation allows the helicity structures $(-+--)$ and $(-+++)$. Compute these amplitudes and show that they are zero.

(f) Compute the nonzero scattering amplitudes contributing to $u\overline{u} \to gg$. Square, sum over final helicities and colors and average over initial helicities and colors. Show that, in the center of mass frame,

$$\frac{d\sigma}{d\cos\theta} = \frac{16\pi\alpha_s^2}{27s}\left(\frac{u}{t} + \frac{t}{u} - \frac{9}{4}\frac{t^2+u^2}{s^2}\right) . \tag{8}$$

(g) For the process $gg \to gg$, the amplitude for $(++++)$, that is, incoming helicity $= +4$, is not obviously zero. Compute this amplitude and show that it is zero.

(h) For the process $gg \to gg$, the amplitude for $(+++-)$, that is, incoming helicity $= +2$, is not obviously zero. Compute this amplitude and show that it is zero.

(i) For the process $gg \to gg$, the amplitude for $(++--)$, that is, incoming helicity $= 0$, is not obviously zero. Compute this amplitude and show that it is nonzero.

(j) Compute the nonzero scattering amplitudes contributing to $gg \to gg$. Actually, you can find the remaining nonzero amplitudes from the answer to part (i) using crossing symmetry. Square, sum over final helicities and colors and average over initial helicities and colors. Show that, in the center of mass frame,

$$\frac{d\sigma}{d\cos\theta} = \frac{9\pi\alpha_s^2}{4s}\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right) . \tag{9}$$

For a swifter approach to these calculations (not to be used in your solution to this problem set!), see M. E. Peskin, arXiv:1101.2414. For deeper insight into gauge theory scattering amplitudes, see H. Elvang and Y.-T. Huang, arXiv:1308.1697 and their book *Scattering Amplitudes in Gauge Theory and Gravity* (Cambridge University Press, 2015).