Physics 331 – Problem Set # 5

(due Wednesday, February 12)

1. In class, we computed the contribution to the effective electric charge from electron-positron loops.

   (a) Graph the evolution of $\alpha^{-1}(Q)$ from $Q = 10$ MeV to $Q = 100$ GeV, assuming that $\alpha = 1/137.036$ at $Q = 0$.

   (b) Include also the contribution from muon loops. Put the corresponding curve for $\alpha^{-1}(Q)$ on the graph.

   (c) Include also the contribution from the τ lepton, and from the quarks u, d, s, c, b. (You can ignore the contributions from the top quark and the W bosons.) Use the values of the quark and lepton masses at: http://pdglive.lbl.gov. Remember that quarks come in 3 colors and have fractional electric charges. Also, use the (non-obvious) fact that, by multiplying the QED quark contributions by 1.05, you include the effects of QCD strong interactions to reasonable accuracy.

   (d) Predict the value of $\alpha^{-1}(Q)$ at $Q = m_Z = 91.188$ GeV.

2. This problem has a relevance to the treatment of infrared divergences in QED that I hope you will see when you get to the end.

   (a) The Poisson distribution is the probability distribution over $\{0, 1, 2, \cdots\}$

   $$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda} \ . \tag{1}$$

   Show that

   $$\sum_{n=0}^{\infty} P(n) = 1 \qquad \langle n \rangle = \sum_{n=0}^{\infty} n\, P(n) = \lambda \tag{2}$$

   (b) Compute $\langle n^2 \rangle$, $\langle n^3 \rangle$ and work out

   $$\left\langle (n-\lambda)^2 \right\rangle \qquad \left\langle (n-\lambda)^3 \right\rangle \tag{3}$$

   Show that, in the limit of large λ, these are consistent with the values for a Gaussian distribution with center at λ and $\sigma^2 = \lambda$.

   (c) Now consider the problem of a classical Klein-Gordon field coupled to a known source $j(x)$ that is nonzero in the interval $t_1 < x^0 < t_2$. Solve the equation

   $$(\partial^2 + m^2)\phi(x) = j(x) \ . \tag{4}$$

   That is, find the Fourier transform of the produced field $\phi(q)$. Represent the solution in real space $\phi(x)$ by a Fourier integral.

(d) Show that, if $j(x) = 0$ for $x^0 > t_2$, then the Fourier transform $j(q)$ behaves at worst as $e^{iq^0 t_2}$ as $\text{Im} q^0 \to \infty$. Use this fact to show that $\phi(x)$ is a solution of the homogeneous Klein-Gordon equation for $t > t_2$. Compute the total energy in this field as an integral over d^3q.

(e) Using a semiclassical picture, compute the number of produced Klein-Gordon bosons as an integral over d^3q.

(f) Using Feynman diagrams, compute the probability for the current $j(x)$ to produce one Klein-Gordon boson. Express the result as an integral over d^3q.

(g) Using Feynman diagrams (which might be disconnected), find the probability for the current $j(x)$ to produce n Klein-Gordon bosons.

(h) Compute the expected total energy production and compare to part (d).

3. This problem gives some exercises in the computation of group theory factors for Lie groups.

(a) In class, we defined, for an irreducible representation r of G,

$$\text{tr}[t_r^a t_r^b] = C(r)\delta^{ab} \quad t^a t^a = C_2(r)\mathbf{1} \ . \tag{5}$$

Show that

$$C(r)d_G = C_2(r)d_r \ , \tag{6}$$

where d_r is the dimension of the representation r and d_G is the number of generators of G, or, equivalently, the dimension of the adjoint representation of G. For all representation matrices, $\text{tr}[t_r^a] = 0$; otherwise G would have a continuous Abelian subgroup.

(b) $SU(N)$ is the group of $N \times N$ unitary matrices of determinant 1, with $(N^2 - 1)$ Hermitian generators. Its fundamental representation N is the space of N-dimensional vectors on which the $N \times N$ matrices act. It is conventional to normalize the generators of $SU(N)$ in the N representation by

$$\text{tr}[t_N^a t_N^b] = \frac{1}{2}\delta^{ab} \tag{7}$$

Write the 3 generators of $SU(2)$ (as 2×2 matrices) and the 8 generators of $SU(3)$ (as 3×3 matrices) normalized to this convention.

(c) Compute $C_2(N)$ in this normalization.

(d) Given a representration r of G acting on vectors ξ_α, there is always a representation $\overline{r}$ of the same dimension that represents the infinitesimal group action on $\xi_\alpha^\dagger$. Show that

$$t_{\overline{r}}^a = -(t_r^a)^T \ . \tag{8}$$

Sometimes, the representations r and $\overline{r}$ are equivalent by unitary transformations. For the fundamental representation of $SU(N)$, this is true for $N = 2$ but not for $N > 2$.

(e) Given representations r_1 and r_2, of dimension d_1 and d_2, the product representation $r_1 \otimes r_2$ is a $d_1 \cdot d_2$-dimensional representation acting on vectors $\Xi_{\alpha\beta}$, where $\alpha = 1, \cdots d_1$ and $\beta = 1, \cdots, d_2$. By acting the group infinitesimally on $\Xi_{\alpha\beta}$, show that

$$t^a_{r_1 \otimes r_2} = t^a_{r_1} \otimes \mathbf{1}_{d_2} + \mathbf{1}_{d_1} \otimes t^a_{r_2} \ , \tag{9}$$

where the first matrix of the product acts on the index α and the second on the index β. Then show that

$$\mathrm{tr}[t^a_{r_1 \otimes r_2} t^a_{r_1 \otimes r_2}] = (C_2(r_1) + C_2(r_2)) d_1 d_2 \ . \tag{10}$$

(f) Such product representations are typically reducible. If the representation R is reducible, it can represented in some basis as block-diagonal, with blocks of size d_j such that

$$d_R = \sum_j d_j \tag{11}$$

We write $R = \oplus_j r_j$. If $r_1 \otimes r_2 = \oplus_j r_j$, show that

$$(C_2(r_1) + C_2(r_2)) d_1 d_2 = \sum_j C_2(j) d_j \ . \tag{12}$$

(g) In $SU(N)$, the product $N \otimes \overline{N}$ contains the trivial representation $U = 1$. This product also contains the adjoint representation G of dimension $(N^2 - 1)$, and this exhausts the states in the product. Then use (12) to prove

$$C_2(G) = N \ . \tag{13}$$

(h) Another trick for computing $C_2(r)$ is to reduce G to an $SU(2)$ subgroup. Show that the N of $SU(N)$ can be thought of as a $2 \oplus 1 \oplus 1 \oplus \cdots$ under an $SU(2)$ subgroup of $SU(N)$. Then it possible to compute $C(r)$ from

$$tr[t^3 t^3] = C(N) \ , \tag{14}$$

where t^3 is the representation matrix for J^3 in the SU(2) subgroup. Show that this recovers $C(N) = \frac{1}{2}$. Use this method to compute $C(G) = N$, which is equivalent to (13).

(i) Two other important representations of $SU(N)$ are the antisymmetric and symmetric products (A and S) of two N's. That is $N \otimes N = A \oplus S$ where A is the antisymmetric combination and S is the symmetric combination. Show that these representations have dimension $N(N-1)/2$, for A, and $N(N+1)/2$, for S. Compute $C(A)$ and $C(S)$ by reducing the N to $SU(2)$ and picking out the antisymmetric and symmetric components of the product of two N's. Use this to compute $C_2(A)$ and $C_2(S)$.

(j) Sometimes, in computing Feynman diagrams for non-Abelian gauge theories, we encounter the product

$$t^a_{r_1} \otimes t^a_{r_2} \tag{15}$$

acting on the product representation $r_1 \otimes r_2$. In particular, this factor appears in the Yang-Mills theory Coulomb potential between particles in the representations r_1 and r_2. Show that, When this representation is reduced to its irreducible components r_j, (15) is diagonalized and its eigenvalues are

$$\frac{1}{2}(C_2(r_j) - C_2(r_1) - C_2(r_2)) \tag{16}$$

Show that, always, at least one of these eigenvalues is negative, so that there is always one mode in which the Coulomb potential is attractive.