Physics 331 – Problem Set # 4

(due Wednesday, February 5)

1. This problem concerns the 1-loop corrections to the photon vertex in QED. We will work with the QED Lagrangian for renormalized perturbation theory,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi \\ &\quad + \delta_2\overline{\psi}\, i\, \partial\!\!\!/ \overline{\psi} - \delta_m\overline{\psi}\psi - \delta_1 e\overline{\psi}\gamma_\mu A^\mu\psi - \frac{\delta_3}{4}(F_{\mu\nu})^2 \ . \end{aligned} \tag{1}$$

The second line contains the four counterterm vertices, which are to be adjusted so that the physical mass of the electron remains equal to m, the normalization of the ψ and A_μ propagators remains equal to 1, and the electron coupling to the photon at $q^2 = 0$ remains equal to 1 at each successive loop order in perturbation theory.

(a) Write out the result derived in class for the dimensionally regularized 1-loop diagram contributing to the electron self-energy $\Sigma(p\!\!\!/)$, in terms of an integral over one Feynman parameter x. Determine the counterterms δm and δ_2 to order α.

(b) Compute the 1-loop vertex diagram in QED, with external momenta p and $p+q$ and momentum q flowing in through the photon vertex, using dimensional regularization. To combine denominators, you will need the Feynman parameter identity

$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz \delta(x+y+z-1)\frac{2!}{[xA+yB+zC]^3} \ , \tag{2}$$

The integral is over the region $0 \leq x, y, z \leq 1$ subject to $x+y+z = 1$. (Actually, the Feynman parameter identity generalizes even further; see Peskin & Schroeder, eq. (6.42).) Arrive at an expression in which the loop momentum is integrated out but two integrals over Feynman parameters remain.

(c) Put the initial and final electron on shell ($p^2 = m^2$) and sandwich the loop integrate between initial and final spinors $\overline{u}(p+q)$ and $u(p)$. Using $(p\!\!\!/ - m)u(p) = 0$ and similarly for $\overline{u}(p+q)$, simplify the expression in (b) and bring it into the form

$$-ie\,\overline{u}(p+q)\Big[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]u(p) \tag{3}$$

Add the δ_1 counterterm and determine δ_1 from the condition that the order-α contribution to $F_1(q^2 = 0)$ is zero. Show that $F_2(q^2)$ is finite in the limit $\epsilon \to 0$. Why should this be true?

(d) Write $d = 4 - 2\epsilon$ and take the limit $\epsilon \to 0$. Show that the $1/\epsilon$ terms in δ_1 and δ_2 are equal.

(e) Show that the finite (order ϵ^0) terms in δ_1 and δ_2 are infrared divergent. Regulate this divergence by adding a small mass μ for the photon.

(f) Show that $\delta_1 = \delta_2$. Part (d) is easy, but this part is more challenging.

2. Particle physicists are now thinking seriously about the existence of a vector boson $\mathcal{A}^\mu$ that couples to matter by

$$\mathcal{L} = Q_\epsilon \overline{\psi} \gamma_\mu \psi \mathcal{A}_\mu \tag{4}$$

where Q_ϵ is a "millicharge": $Q_\epsilon = \epsilon e$, where $(-e)$ is the standard charge of an electron and ϵ is a small parameter. For this problem: $m_e = 0.51$ MeV, $m_\mu = 106$ MeV.

(a) Compute the contribution of the 1-loop diagram involving this particle to the $(g-2)$ of the electron and the muon. Use the method that you used in Problem 1, just changing what needs to be changed.

(b) Assume that the mass M of the $\mathcal{A}$ boson is of the order of a few hundred MeV, i.e., $M \gg m_e$, $M \sim m_\mu$. Work out the shifts of $(g-2)$ for the electron and for the muon explicitly for an $\mathcal{A}$ mass in this region.

(c) The current experimental value of the muon $(g-2)$ is greater than the prediction of the Standard Model by

$$a_\mu - a_{\mu,SM} = 3 \times 10^{-9} \; . \tag{5}$$

This is about a 3.5 σ effect, that is, it needs confirmation. Nevertheless, we can make a theory of this difference by attributing it to the effect of $\mathcal{A}$. Compute the required value of ϵ for some values of M in the range 100 MeV – 1 GeV. (Better, plot the curve.)

(d) For the values of (M, ϵ) in part (b), estimate the effect of $\mathcal{A}$ on the electron $(g-2)$. Is the shift greater than the current experimental error

$$\Delta a_e = 2.6 \times 10^{-13} \qquad ? \tag{6}$$