Physics 331 – Problem Set # 3

(due Wednesday, January 29)

1. Consider ϕ^3 theory with three fields, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m_2^2\phi^2 + \frac{1}{2}(\partial_\mu \phi_3)^2 - \frac{1}{2}m_3^2\phi_3^2 - \lambda\phi_1\phi_2\phi_3 \quad (1)$$

This theory does not actually make sense; its potential

$$V = \lambda\phi_1\phi_2\phi_3 \quad (2)$$

is not bounded below. However, it is a useful toy model to study in perturbation theory.

(a) Assume that $m_1 < m_2 < m_3$. Compute the the propagator of the field ϕ_1 including 1-loop diagrams. This is of the form

$$\frac{i}{p^2 - m_1^2 - \Pi(p^2) + i\epsilon} \quad (3)$$

where $-i\Pi(p^2)$ is the sum of 1-particle-irreducible diagrams. In 1-loop order, there is only 1 Feynman diagram. Compute $\Pi(p^2)$ to this order as a Feynman parameter integral. You will find that the integral over the loop momentum is divergent. Compute this integral using dimensional regularization. Show that the propagagator can be made finite by the replacement

$$m_1^2 - \frac{\lambda^2}{(4\pi)^2}[\frac{1}{\epsilon} - \gamma + \log 4\pi] \to m_{1R}^2 \ , \quad (4)$$

using $d = 4 - 2\epsilon$. This is a replacement of the Langrangian parameter m_1^2 with a value m_{1R}^2 that is closer to the physical mass of the ϕ_1 particle. Show that $\Pi(p^2)$ is real as long as $p^2 < m_2^2 < m_3^2$.

(b) Evaluate the Feynman parameter integral explicitly and compute the shift from m_{1R} to the physical ϕ_1 mass M_1, equal to $(M_1 - m_{1R})$, to order λ^2.

(c) Notice that, at large values of p^2, the logarithm has a branch cut along which $\Pi(p^2)$ has an imaginary part. Find the value of p^2 where the branch cut begins. Since $\Pi(p^2)$ is real on the real p^2 axis below this point, analytic continuation implies that this function must satisfy $\Pi(a + ib) = (\Pi(a - ib))^*$. Show that the imaginary part of $\Pi(p^2)$ along the cut is negative at $p^2 + i\epsilon$ and positive at $p^2 - i\epsilon$.

(d) Now consider the case $m_2 < m_3$, $m_1 > (m_2 + m_3)$, in which the ϕ_1 meson is unstable and can decay to $\phi_2\phi_3$. Compute the decay width for $\phi_1 \to \phi_2\phi_3$ to leading order in λ.

(e) Evaluate $\Pi(p^2)$ for this case at $p^2 = M_1^2 + i\epsilon$ (as the propagator indicates) and in particular find its imaginary part. Show that the full propagator has a pole in the complex plane at a negative imaginary value near p^2. Show that, near the pole, the propagator has the Breit-Wigner form

$$\frac{iZ}{p^0 - M_1 + i\Gamma/2} \tag{5}$$

where Z is a constant, M_1 is the physical mass, and Γ is the decay width computed in part (d).

(f) (Extra credit) How is the presence of this pole compatible with the requirement $\Pi(a+ib) = (\Pi(a-ib))^*$? Show that the pole is actually on the second Riemann sheet going through the the branch cut from above. Show that there is actually another pole on the other second sheet going through the branch cut from below. Why is this latter pole irrelevant?

2. Consider ϕ^4 theory in 4 dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \ . \tag{6}$$

Compute the scattering amplitude for $\phi\phi \to \phi\phi$ to order λ^2.

(a) Show that, in leading order

$$i\mathcal{M}(\phi(p)\phi(q) \to \phi(k)\phi(\ell)) = -i\lambda_0 \tag{7}$$

independent of the momenta p, q, k, ℓ.

(b) Draw the ϕ self-energy diagram of order λ_0^1. Show that this diagram, whatever its value might be, is entirely accounted by shifting the mass of the ϕ from the bare value m_0 to the physical value m.

(c) Draw the three diagrams that contribute in order λ_0^2. Notice that these diagrams are logarithmically divergent. In computing these diagrams, use the physical mass m.

(d) The three diagrams involve the same Feynman integral. Compute this integral using dimensional regularization: expand in $\epsilon = 2 - d/2$, keep the terms of order ϵ^{-1} and ϵ^0, then evaluate the integral over the Feynman parameter for each term.

(e) Write a dimensionally regularized expression for the scattering amplitude to order λ_0^2. This is conveniently written as a function of s, t, and u.

(f) Let the physical value of the coupling constant λ be given by the value of the scattering amplitude at threshold: $s = 4m^2$, $t = u = 0$. This quantity is potentially measurable. Solve for λ_0 in terms of λ.

(g) Write the answer to part (e) as a function of λ. Note that the expression for the scattering amplitude is now finite when expressed in terms of the measurable physical mass and coupling parameter.

(h) Show that the expression in part (g) satisfies the optical theorem,

$$\mathrm{Im}[\mathcal{M}(s, t = 0)] = \sqrt{s(s - 4m^2)}\ \sigma(s)\ , \tag{8}$$

to order λ^2. The imaginary part of $\mathcal{M}$ arises from the imaginary part of the logarithm, evaluated for $s > 4m^2$ above the cut (at $s + i\epsilon$).