Physics 331 – Problem Set # 2

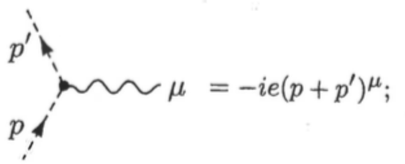
(due Wednesday, January 22)

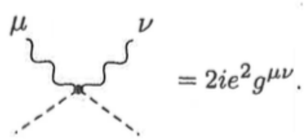
1. Consider a complex-valued scalar field coupled to electromagnetism. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi \ , \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu \phi$ is the covariant derivative.

(a) Work out the Feynman rules for the coupling of photons to the scalar particle. You should find





(b) Compute the differential and total cross section, at leading order, for $e^+e^- \to \phi\phi^*$. Ignore the electron mass, but do not ignore the mass m of the scalar particle. Compare your answer to the result for $e^+e^- \to \mu^+\mu^-$. You should find a different value for the total cross section and a different shape of the angular distribution.

(c) Derive the Schwinger-Dyson equation–the partial differential equation satisfied by the Green's function—for $\langle \phi(x) \phi^*(y) \rangle$.

(d) Show that the electromagnetic current in this theory—that is, the current that appears on the right-hand side of Maxwell's equations—is given by

$$iej^\mu(x) = \delta \mathcal{L} / \delta A_\mu(x) \tag{2}$$

Work out the form of $j^\mu(x)$ for this theory. Note that it includes a term proportional to A_μ. Show that this term is needed in order for the classical equations of motion to imply that this current is conserved.

(e) Draw the Feynman diagrams contributing to $\phi\phi^* \to \gamma\gamma$ at the leading order in perturbation theory, and construct the scattering amplitude in the form

$$i\mathcal{M}(\phi(p)\phi^*(\overline{p}) \to \gamma(k_1)\gamma(k_2)) = i\mathcal{M}^{\mu\nu}(p, q, k_1, k_2)\ \epsilon_\mu(k_1)\ \epsilon_\nu(k_2) \tag{3}$$

with $p + \overline{p} - k_1 - k_2 = 0$. For this amplitude, the Ward identity implies that, if $p^2 = \overline{p}^2 = m^2$, then the substitution $\epsilon^\mu(k_1) \to k_1^\mu$ causes the amplitude to vanish. Show this explicitly for the leading-order diagrams. The diagram with the 4-point vertex is essential for this cancellation.

2. When the A_μ field in electrodynamics is quantized with the gauge parameter ξ, the gauge-fixed Lagrangian of QED can be written

$$\mathcal{L} = \frac{1}{2}A_\mu(\partial^2 g^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu)A_\nu + \bar{c}(-\partial^2)c + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \ . \qquad (4)$$

where c, $\bar{c}$ are fermionic fields, the ghost fields. This Lagrangian has an unexpected symmetry.

(a) Show that the Lagrangian in (4) is equivalent to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + B\partial_\mu A^\mu + \frac{\xi}{2}B^2 + \bar{c}(-\partial^2)c + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \qquad (5)$$

by writing the functional integral for (5) and integrating out the "auxiliary field" $B(x)$.

(b) Show that the Lagrangian (5) is invariant under the transformation

$$\begin{aligned} \delta A_\mu &= \epsilon\partial_\mu c \\ \delta\psi &= -ie\epsilon c\psi \\ \delta c &= 0 \\ \delta\bar{c} &= \epsilon B \\ \delta B &= 0 \end{aligned} \qquad (6)$$

where ϵ is a Grassmann number. This is called the BRST (Becchi-Rouet-Stora-Tyutin) transformation. Note that the (anticommuting) ghost field appears as the parameter of a local gauge transformation.

(c) Show that the BRST transformation is nilpotent: δ^2 on any field gives 0.

The Ward identities for a non-Abelian gauge theory are quite complicated. Those of BRST symmetry (which generalizes to the non-Abelian case) are equivalent and simpler. The nilpotent property of BRST is important in proving that the gauge-fixed theory is unitary.