Physics 331 – Problem Set # 1

(due Wednesday, January 15)

1. Consider the zero-dimensional functional integral

$$Z = \int_{-\infty}^{\infty} d\phi \ \exp[-\phi^2/2A] \tag{1}$$

as a model for a general functional integral.

(a) Compute $\langle \phi^n \rangle$, given by

$$\langle \phi^n \rangle = \left(\int_{-\infty}^{\infty} d\phi \ \phi^n \ \exp[-\phi^2/2A] \right) \ / \ Z \ . \tag{2}$$

This should be of the form

$$\langle \phi^n \rangle = \mathcal{N}(n) \ A^{n/2} \ . \tag{3}$$

Find $\mathcal{N}(n)$.

(b) Define the *contraction* of ϕ's to be

$$[\phi \ \phi] = \left\langle \phi^2 \right\rangle \ . \tag{4}$$

Compute the contraction (4). Show that $\mathcal{N}(n)$ in (3) equals the number of contractions expected from Wick's Theorem. Then the result of part (a) is just Wick's Theorem for this "quantum" system.

(c) Show that the same result holds for the integral over many (real) variables ϕ_i. Consider

$$Z = \int_{-\infty}^{\infty} d^n\phi \ \exp[-\frac{1}{2}\phi_i A^{-1}_{ij} \phi_j] \ , \tag{5}$$

where the matrix A is real, symmetric, and invertable. Using this measure, show that

$$\langle \phi_i \phi_j \rangle = A_{ij} \ , \tag{6}$$

and define this to be the contraction. Then show that the general expectation value $\langle \phi_i \phi_j \cdots \phi_k \rangle$ is equal to the sum of all possible contractions. [Hint: A is a real symmetric matrix; diagonalize it.]

2. The statistical mechanics partition function

$$Z = \ tr[e^{-\beta H}] \tag{7}$$

can be computed as a functional integral over configurations $\{q_i(\tau)\}$, where τ is Euclidean time, $t = -i\tau$, and the configurations are periodic in τ with period β. Here are some examples:

(a) The Euclidean Lagrangian for a simple harmonic oscillation with $m = 1$ is

$$L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 \tag{8}$$

Evaluate the functional integral over $x(\tau)$ by introducing a Fourier decomposition

$$x(\tau) = \sum_n x_n \frac{1}{\sqrt{\beta}}\ e^{2\pi i n\tau/\beta} \tag{9}$$

and integrating over the Fourier modes.

(b) It is tricky to assess the dependence of your answer on β, since the functional measure can also contain factors of β in a way that is not so obvious. However the dependence on ω comes only through (8) and therefore should be unambiguous. Compute the partition function as a function of ω in the standard way. You should find

$$Z = e^{-\beta\omega/2}[1 - e^{-\beta\omega}]^{-1}\ . \tag{10}$$

Now show that the functional integral evaluation in (a) reproduces this result, up to multiplication by an overall β-dependent constant.

$$\text{Hint:} \qquad \sinh z = z \cdot \prod_{n=1}^{\infty}(1 + \frac{z^2}{(n\pi)^2}) \tag{11}$$

(c) Generalize this construction to a free scalar field with mass m. Compute the partition function in the standard way as a function of the mass m. Then compute the functional integral over fields in Euclidean space periodic in τ with period β,

$$\int \mathcal{D}\phi\ \exp\Big[-\int d\tau d^3x\ \frac{1}{2}((\partial_\mu\phi)^2 + m^2\phi^2)\Big] = (\det(-\partial^2 + m^2))^{-1/2} \tag{12}$$

and show that the dependence on m is the same.

(d) Let $\psi(\tau)$, $\overline{\psi}(\tau)$ be Grassman-valued coordinates, with the Euclidean Lagrangian

$$L_E = \overline{\psi}\dot{\psi} + \omega\overline{\psi}\psi\ . \tag{13}$$

By thinking about the interpretation of the correlation function $\left\langle \psi(\tau_1)\overline{\psi}(\tau_2)\right\rangle$ in terms of a time-ordered product of operators, argue that the trace over states is given by a functional integral for $\psi(\tau)$, $\overline{\psi}(\tau)$ in which these fields have antiperiodic boundary conditions $\psi(\tau + \beta) = -\psi(\tau)$.

(e) The Lagrangian (13) corresponds to the Hamiltonian

$$H = \omega\overline{\psi}\psi\ , \tag{14}$$

that is, to a simple 2-level system. Compute the partition function of this system in the standard way and by functional integration, and show that the dependence of the results on ω is the same.