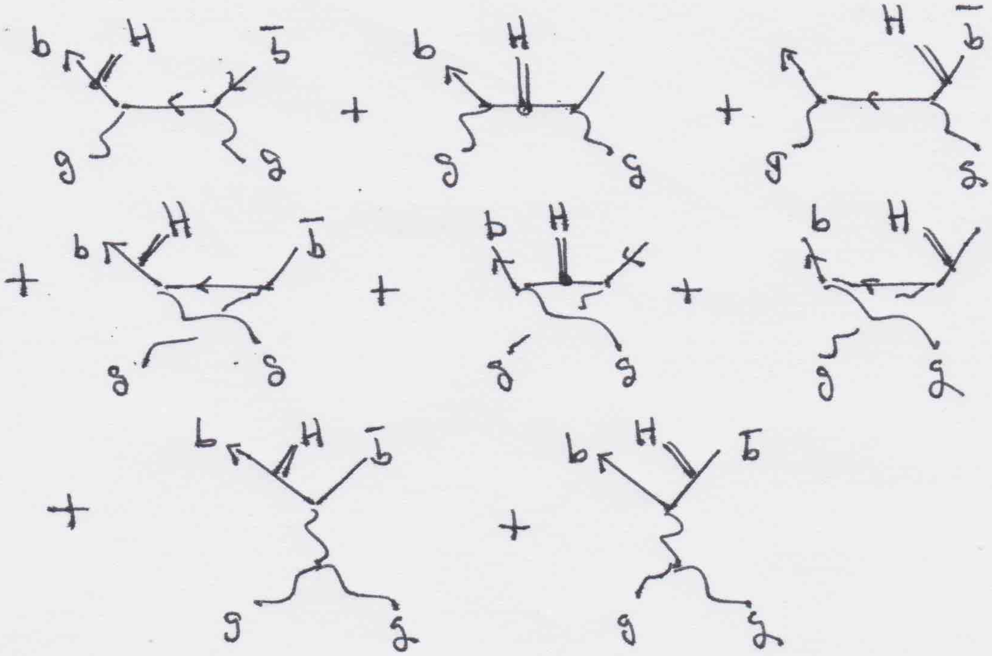
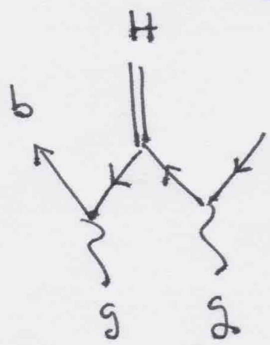
Physics 331 – Final Exam

Solutions

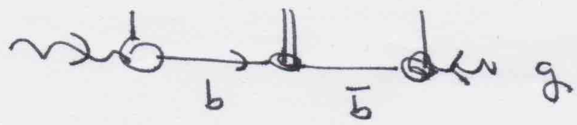
1.) a.)



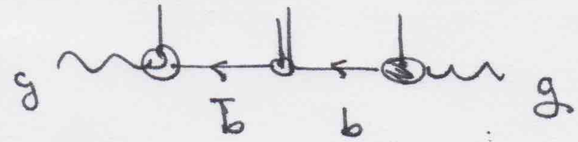
b.) the log enhancements are generated by collinear splitting. The doubly enhanced diagram is



The internal $b, \bar{b}$ lines have momenta almost collinear with the initial g's. (Actually, there are 2 contributions!)



and



In all other diagrams, at least one propagator is off-shell by an amount $\sim m_H^2$.

c) $$\sigma(b\bar{b} \to H) = \frac{1}{2s} \int \frac{d^3 p_H}{(2\pi)^3} \frac{1}{2E_H} (2\pi)^4 \delta^{(4)}(p_b + p_{\bar{b}} - p_H) \, |\mathcal{M}|^2$$

$$= \frac{1}{2s} \int \frac{d^4 p_H}{(2\pi)^4} \, 2\pi \delta(p_H^2 - m_H^2) \, (2\pi)^4 \delta(p_b + p_{\bar{b}} - p_H) \, |\mathcal{M}|^2$$

$$= \frac{\pi}{s} \, \delta(s - m_H^2) \, |\mathcal{M}|^2$$

It is a good approximation to ignore the b quark mass. Then

$$\Delta \mathcal{L} = Y H (b_L^\dagger b_R + b_R^\dagger b_L)$$

for b_L: $\quad u(p_b) = \sqrt{2E_b} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad p_b \parallel \hat{3}$

$\bar{b}_L$: $\quad v(p_{\bar{b}}) = \sqrt{2E_b} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad p_{\bar{b}} \parallel -\hat{3}$

$\mathcal{M}(b_L \bar{b}_L \to H) =$ $\perp$ $= (-2E_b)(iY) \qquad |\mathcal{M}|^2 = s \cdot Y^2$

$\mathcal{M}(b_R \bar{b}_R \to H) =$ same answer, by $\underline{P}$.

avg over spins $\quad \frac{1}{2}\cdot\frac{1}{2}\cdot 2\cdot|\mathcal{M}(b_L\bar{b}_L\to H)|^2$

avg over colors $\quad \frac{1}{3}\cdot\frac{1}{3}\cdot 3$

in all

sum/avg over spins + colors: $\quad \frac{1}{4}\cdot\frac{1}{9}\cdot\sum|\mathcal{M}|^2 = \frac{1}{6}m_H^2 Y^2$

$$\sigma(b\bar{b}\to H) = \frac{1}{6}\pi Y^2\,\delta(s-m_H^2)$$

d.) Collinear splitting of each gluon gives

$$\sigma(g+X\to Y) \cong \frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{k_T^2}\int_0^1 dx\; P_{b\leftarrow g}(x)\cdot\sigma(b+X\to Y)\Big|_{\hat{s}=xs}$$

We can estimate the limits of the dk_T^2 integral as

$$\int_{m_b^2}^{m_H^2} dk_T^2$$

m_H^2 ← b is no longer collinear

m_b^2 ← cannot ignore b mass

$$\sigma(g+X\to Y) \cong \frac{\alpha_s}{2\pi}\log\frac{m_H^2}{m_b^2}\int_0^1 dx\;\frac{1}{2}[x^2+(1-x)^2]\cdot\sigma(b+X\to Y)\Big|_{\hat{s}=xs}$$

The enhancement factor is

$$\log\left(\frac{m_H^2}{m_b^2}\right) \approx \log\left(\frac{500\text{ GeV}}{4\text{ GeV}}\right)^2 = 9.7$$

The doubly-enhanced term is

$$\sigma(gg \to H) = \left(\frac{\alpha_s}{2\pi} \log \frac{m_H^2}{m_b^2}\right)^2 Y^2$$

$$\cdot \int_0^1 dx_1 \ \tfrac{1}{2}\left[x_1^2 + (1-x_1)^2\right] \int_0^1 dx_2 \ \tfrac{1}{2}\left[x_2^2 + (1-x_2)^2\right] \cdot \frac{\pi}{6} Y^2 \delta(x_1 x_2 s - m_H^2)$$

$$\times 2 \quad (\text{for } b\bar{b} \to H, \ \bar{b}b \to H)$$

Integrate x_2 over the δ-function: $x_2 = \dfrac{m_H^2}{x_1 s}$

$$\sigma(gg \to H) = \frac{\alpha_s^2}{48\pi} \log^2 \frac{m_H^2}{m_b^2} Y^2$$

$$\int_{m_H^2/s}^1 dx_1 \ \frac{1}{x_1 s}\left[x_1^2 + (1-x_1)^2\right]\left[\left(\frac{m_H^2}{x_1 s}\right)^2 + \left(1 - \frac{m_H^2}{x_1 s}\right)^2\right]$$

The integral is elementary; its value is $a = m_H^2/s$

$$\frac{1}{s} \cdot f(a) = \frac{1}{s}\left[(1+2a)^2 \log \frac{1}{a} - 2(1+2a-3a^2)\right]$$

Then

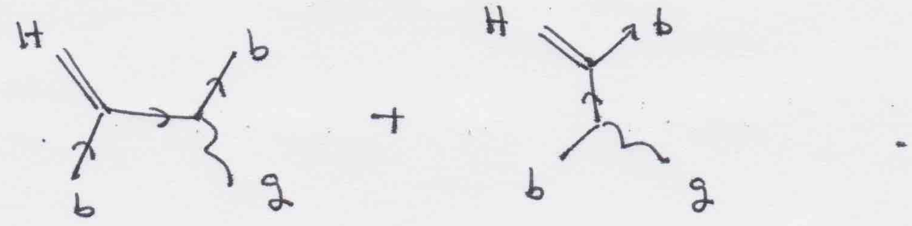
$$\sigma(gg \to H) = \frac{\alpha_s^2 Y^2}{48\pi s}\left(\log \frac{m_H^2}{m_b^2}\right)^2 f(a)$$

e.) for $m_H = 500$ GeV, $\sqrt{s} = 1400$ GeV, $f(a) = 0.83$

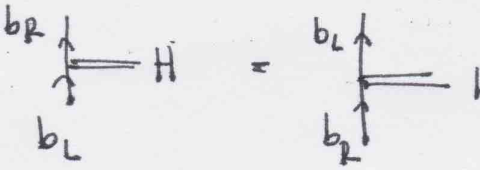
using $(\hbar c)^2 = 0.389\ (\mathrm{GeV})^2$-mb $\quad \alpha_s \simeq 0.12$

then

$$\sigma(gg \to H) = 1.5 \times 10^{-9}\,\text{mb} = 1.5\ \text{pb}$$

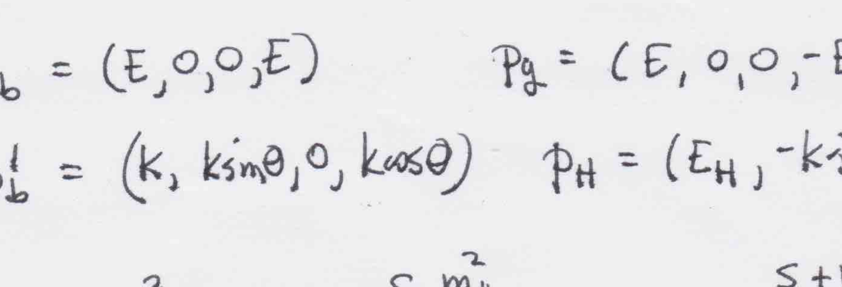
f.) 

g.) To compute the diagrams, it is easiest to work with states of definite helicity. Note that the Hbb coupling flips the helicity



$= iY$

work in the CM frame; ignore the b quark mass



$p_b = (E, 0, 0, E)$ $\qquad p_g = (E, 0, 0, -E)$

$p_b' = (k, k\sin\theta, 0, k\cos\theta)$ $\qquad p_H = (E_H, -k\sin\theta, 0, -k\cos\theta)$

$$s = 4E^2 \qquad k = \frac{s - m_H^2}{2\sqrt{s}} \qquad E_H = \frac{s + m_H^2}{2\sqrt{s}}$$

The Feynman diagrams give

$$i\mathcal{M} = (ig_s t^a)(i Y)$$

$$\bar{u}(p_b')\left[\gamma\cdot\varepsilon(p_g)\,\frac{i\gamma(p_b'-p_g)}{(p_b'-p_g)^2} + \frac{i\gamma\cdot(p_b+p_g)}{(p_b+p_g)^2}\,\gamma\cdot\varepsilon(p_g)\right]u(p_b)$$

It is easy to check that this expression satisfies the Ward Identity

$$p_g^\mu \mathcal{M}_\mu = 0$$

Write $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ and decompose $u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$

Let's first compute the matrix elements for

$$b_L g_R \to b_R H, \qquad b_L g_L \to b_R H$$

The matrix elements for $b_R g_L \to b_L H$, $b_R g_R \to b_L H$ will be equal by Parity. The needed 2-component spinors are

$$u_L(p_b) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u_R(p_b') = \sqrt{2k}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

Then the above becomes.

$$i\mathcal{M} = (-ig_s t^a Y)\sqrt{4Ek}$$

$$\cdot(\cos\theta/2, \sin\theta/2)\left[\sigma\cdot\varepsilon(p_g)\,\frac{\bar{\sigma}\cdot(p_b'-p_g)}{-2p_b'\cdot p_g} + \frac{\sigma\cdot(p_b+p_g)}{+2p_b\cdot p_g}\,\bar{\sigma}\cdot\varepsilon(p_g)\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for g_R $\quad \varepsilon(p_g) = \frac{1}{\sqrt{2}}(0,-1,i,0)$ $\quad$ since $p_g \parallel -\hat{3}$

$$\sigma\cdot\varepsilon = -\vec{\sigma}\cdot\vec{\varepsilon} = \begin{pmatrix}0&0\\-\sqrt{2}&0\end{pmatrix} \qquad \bar{\sigma}\cdot\varepsilon = +\vec{\sigma}\cdot\vec{\varepsilon} = \begin{pmatrix}0&0\\\sqrt{2}&0\end{pmatrix}$$

for g_L $\quad \varepsilon(p_g) = \frac{1}{\sqrt{2}}(0,-1,-i,0)$

$$\sigma\cdot\varepsilon = -\vec{\sigma}\cdot\vec{\varepsilon} = \begin{pmatrix}0&-\sqrt{2}\\0&0\end{pmatrix} \qquad \bar{\sigma}\cdot\varepsilon = +\vec{\sigma}\cdot\varepsilon = \begin{pmatrix}0&\sqrt{2}\\0&0\end{pmatrix}$$

$$\bar{\sigma}\cdot(p_b'-p_g) = \begin{bmatrix}k-E+(k\cos\theta+E) & k\sin\theta\\ k\sin\theta & (k-E)-(k\cos\theta+E)\end{bmatrix}$$

$$\sigma\cdot(p_b+p_g) = \sigma\cdot(2E,0,0,0) = 2E\cdot\underline{1}$$

then for $b_L g_R \to b_R H$

$$i\mathcal{M} = (-ig_s t^a Y)\sqrt{4Ek}$$

$$(\cos\theta/2, \sin\theta/2)\left[\begin{pmatrix}0&0\\-\sqrt{2}&0\end{pmatrix}\begin{pmatrix}\otimes & k\sin\theta\\ \otimes & \otimes\end{pmatrix}\frac{1}{(-2Ek(1+\cos\theta))} + \frac{2E}{(2E)^2}\underbrace{\begin{pmatrix}0&0\\\sqrt{2}&0\end{pmatrix}\Big]\begin{pmatrix}0\\1\end{pmatrix}}_{0}\right.$$

$$= -ig_s t^a Y \sin\theta/2 \;\sqrt{2}\;\frac{k\sin\theta}{2Ek(1+\cos\theta)}\sqrt{4Ek}$$

$$\doteq -ig_s t^a Y\sqrt{2}\,\frac{k}{\sqrt{Ek}}\;\frac{\sin\theta/2\;\sin\theta}{(1+\cos\theta)}$$

$$= -ig_s t^a Y\sqrt{2}\,\frac{k}{\sqrt{Ek}}\;\sin\theta/2\;\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$= -ig_s t^a Y\,\frac{\sqrt{2}\,k}{\sqrt{Ek}}\;\frac{\sin^2\theta/2}{\cos\theta/2}$$

for $b_L g_L \to b_R H$

$$i\mathcal{M} = (-ig_s t^a Y)\sqrt{4Ek}$$

$$(\cos\theta/2, \sin\theta/2)\left[\begin{pmatrix}0&-\sqrt{2}\\0&0\end{pmatrix}\begin{bmatrix}\otimes & \otimes\\ \otimes & k(1-\cos\theta)-2E\end{bmatrix}\frac{1}{(-2Ek(1+\cos\theta))} + \frac{1}{2E}\begin{pmatrix}0&\sqrt{2}\\0&0\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -i g_s t^a Y \cos\theta/2 \; \frac{\sqrt{4Ek}\;\sqrt{2}}{2Ek(1+\cos\theta)} \left[k(1-\cos\theta) - 2E + k(1+\cos\theta) \right]$$

$$= -i g\, t^a\, Y \frac{\sqrt{2}}{\sqrt{EK}} \; \frac{\cos\theta/2}{2\cos^2\theta/2} \left[2(k-E) \right]$$

$$= +i g\, t^a\, Y \sqrt{2}\; \frac{1}{\sqrt{EK}} (E-k) \frac{1}{\cos\theta/2}$$

in all

$$i\mathcal{M} = i g_s t^a Y \frac{\sqrt{2}}{\sqrt{EK}} \cdot \begin{cases} -k \sin^2\theta/2 & b_L g_R \\ (E-k) & b_L g_L \end{cases} \cdot \frac{1}{\cos\theta/2}$$

$$|\mathcal{M}|^2 = (t^a t^a)\; 2 g_s^2 Y^2 \frac{1}{EK} \cdot \frac{1}{\cos^2\theta/2} \begin{cases} k^2 \sin^4\theta/2 & b_L g_R \\ (E-k)^2 & b_L g_L \end{cases}$$

There are interesting things to say about these expressions

① The $b_L g_L$ initial state has $J^3 = +\tfrac{1}{2}$

$$b_L \xrightarrow{\;\leftarrow\;} \quad \overset{\longrightarrow}{\leftsquigarrow} \; g_L$$

so it is easy for the final b_R to go forward: $H \Leftarrow \quad \overset{\longrightarrow}{\longrightarrow} b_R$

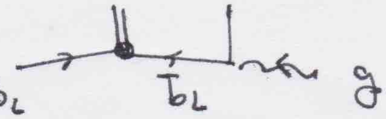
But, the $b_L g_R$ initial state has $J^3 = -3/2$

$$b_L \xrightarrow{\;\leftarrow\;} \quad \overset{\longleftarrow}{\leftsquigarrow} \; g_R$$

so forward scattering requires $L = 2$. This is the origin of the factor $\sin^2\theta/2$ in the amplitude.

② The first diagram on p. 5 has a collinear singularity

The collinear kinematics is



this is in the backward direction $\cos\theta \Rightarrow -1$ or $\cos\theta/2 \to 0$

If

$$p(\bar{b}_L) \sim x\, p_g$$

then $\quad s(b\bar{b}) = m_H^2 \Rightarrow \quad x = m_H^2/s$

The spin-dependent splitting functions are

$$g_R \to \bar{b}_L \qquad P(x) \sim (1-x)^2 \sim \left(\frac{s-m_H^2}{s}\right)^2 \sim k^2$$

$$g_L \to \bar{b}_L \qquad P(x) \sim x^2 \sim \left(\frac{m_H^2}{s}\right)^2 \sim (E-k)^2$$

since $\quad k = \dfrac{s-m_H^2}{2\sqrt{s}} = \dfrac{4E^2 - m_H^2}{4E} \qquad E-k = \dfrac{m_H^2}{2\sqrt{s}}$.

This explain the difference between the two amplitudes.

Now we need to assemble the cross-section. The color average/sum is

$$\frac{1}{3}\cdot\frac{1}{8}\, \mathrm{tr}\, t^a t^a = \frac{1}{3}\,\frac{1}{8}\,\frac{1}{2}\,\delta^{aa} = \frac{1}{6}\,\frac{1}{8}\cdot 8 = \frac{1}{6}$$

The spin average/sum is

$$\frac{1}{2}\frac{1}{2}\cdot 2\left(|\mathcal{M}(b_L g_R)|^2 + |\mathcal{M}(b_L g_L)|^2\right)$$

$$= \frac{1}{2}\left(|\mathcal{M}(b_L g_R)|^2 + |\mathcal{M}(b_L g_L)|^2\right)$$

Then the spin avg/sum color avg/sum squared amplitude is

$$\frac{1}{12}\; 2 g_s^2 Y^2 \frac{1}{(Ek)^2}\,\frac{2}{(1+\cos\theta)}\left[\frac{k^2(1-\cos\theta)^2}{4} + (E-k)^2\right]$$

Finally we need to put the phase space into the correct form

$$\int d\Pi_2 = \frac{1}{8\pi}\int_{-1}^{1}\frac{d\cos\theta}{2}\ \frac{k}{E} = \frac{1}{16}\ \frac{k}{E}\int d\cos\theta$$

We are working in the CM frame where k, E have fixed values. In this frame, define

$$k_{\parallel} = (k^2 - k_T^2)^{1/2}$$

Actually, for each k_T there are 2 values of the parallel momentum, $+k_{\parallel}$ and $-k_{\parallel}$.

$$k_T = k\sin\theta = k(1-\cos^2\theta)^{1/2}$$

$$\frac{dk_T}{d\cos\theta} = k\frac{\cos\theta}{\sin\theta} \qquad \frac{d\cos\theta}{dk_T} = \frac{1}{k}\left|\frac{\sin\theta}{\cos\theta}\right| = \frac{1}{k}\ \frac{k_T}{k_{\parallel}}$$

Then

$$\int d\Pi_2 = \frac{1}{16\pi}\int_0^k dk_T\ \frac{k_T}{E\ k_{\parallel}}$$

but we must include the integral for both solutions $\pm\cos\theta$.

There is a singularity at $k_T = k$, call the Jacobian peak

$$\int^k \frac{dk_T}{(k^2 - k_T^2)^{1/2}}$$

This singularity is integrable.

Putting all of the pieces together

$$\sigma = \frac{1}{2s}\ \frac{1}{16\pi}\int_0^k dk_T\ \frac{k_T}{Ek_{\parallel}}\ \frac{1}{12}\ g_s^2 Y^2\ \frac{1}{Ek}$$

$$\left\{ \frac{k}{k+k_{\parallel}}\left[(k-k_{\parallel})^2 + 4(E-k)^2\right] + \frac{k}{k-k_{\parallel}}\left[(k+k_{\parallel})^2 + 4(E-k)^2\right] \right\}$$

then with $E - k = \frac{m_H^2}{2\sqrt{s}} = \frac{m_H^2}{4E}$

$$\frac{d\sigma}{dk_T} = \frac{\alpha_s}{96 s} Y^2 \frac{k_T}{E^2 k_\parallel} \left\{ \frac{1}{k+k_\parallel}\left((k-k_\parallel)^2 + \frac{m_H^4}{4E^2}\right) + \frac{1}{k-k_\parallel}\left((k+k_\parallel)^2 + \frac{m_H^4}{4E^2}\right) \right\}$$

$$\frac{d\sigma}{dk_T} = \frac{\alpha_s}{24} Y^2 \frac{1}{s^2} \frac{k_T}{k_\parallel} \left\{ \frac{1}{k+k_\parallel}\left((k-k_\parallel)^2 + \frac{m_H^4}{s}\right) + \frac{1}{k-k_\parallel}\left((k+k_\parallel)^2 + \frac{m_H^4}{s}\right) \right\}$$

Here $k, E, k_\parallel$ are evaluated in the CM frame. But, then, this expression is correct in any frame boosted w. respect to the CM frame along the bg axis $b \rightarrow \leftarrow g$

b) The b quark comes from splitting the other gluon. The term is $gg \to H + b$ enhanced by one log is

$$\frac{d\sigma}{dk_T} = \frac{\alpha_s}{2\pi} \log\frac{m_H^2}{m_b^2} \int_0^1 dx \; \frac{1}{2}\left[x^2 + (1-x)^2\right]$$

$$\cdot \left[\frac{\alpha_s}{24 s^2} Y^2 \frac{k_T}{k_\parallel} \left\{ \frac{1}{(k+k_\parallel)}\left((k-k_\parallel)^2 + \frac{m_H^4}{s}\right) + \frac{1}{k-k_\parallel}\left((k+k_\parallel)^2 + \frac{m_H^4}{s}\right) \right\} \right]_{s = x s(gg)}$$

The term in brackets is to be evaluated at $s = x \cdot s(gg)$. There is also a process

g splitting to $\bar{b}$, $\bar{b} + g \to H + \bar{b}$

which produces $H + \bar{b}$ jet. The $\bar{b}$ jet is not easily distinguished from a b jet, so this multiplies the rate by a factor of 2.

Finally, we find

$$\frac{d\sigma}{dk_T}(gg \to H + b\text{-jet})$$

$$\cong \frac{\alpha_s^2}{24\pi} Y^2 \left(\log\frac{m_H^2}{m_b^2}\right) \int_0^1 dx \; \tfrac{1}{2}[x^2 + (1-x)^2]$$

$$\left[\frac{1}{s^2} \frac{k_T}{k_{\parallel}} \left\{ \frac{1}{k+k_{\parallel}}\left((k-k_{\parallel})^2 + \frac{m_H^4}{s}\right) + \frac{1}{k-k_{\parallel}}\left((k+k_{\parallel})^2 + \frac{m_H^4}{s}\right)\right] \right._{s = x\, s(gg)}$$

ii.) It is not so hard to do the integral numerically, but, for the problem set, I asked you to evaluate x at $x = \frac{1}{2}$. Then replace

$$\int_0^1 dx \, \tfrac{1}{2}[x^2 + (1-x)^2] \, f(x) \;\to\; \tfrac{1}{3} f(x = \tfrac{1}{2})$$

Note that this gives $s = \frac{1}{2} s(gg) = (990\text{ GeV})^2$, not $(700\text{ GeV})^2$ as it says in the problem set.

$$\frac{d\sigma}{dk_T}(gg \to H + b\text{-jet})$$

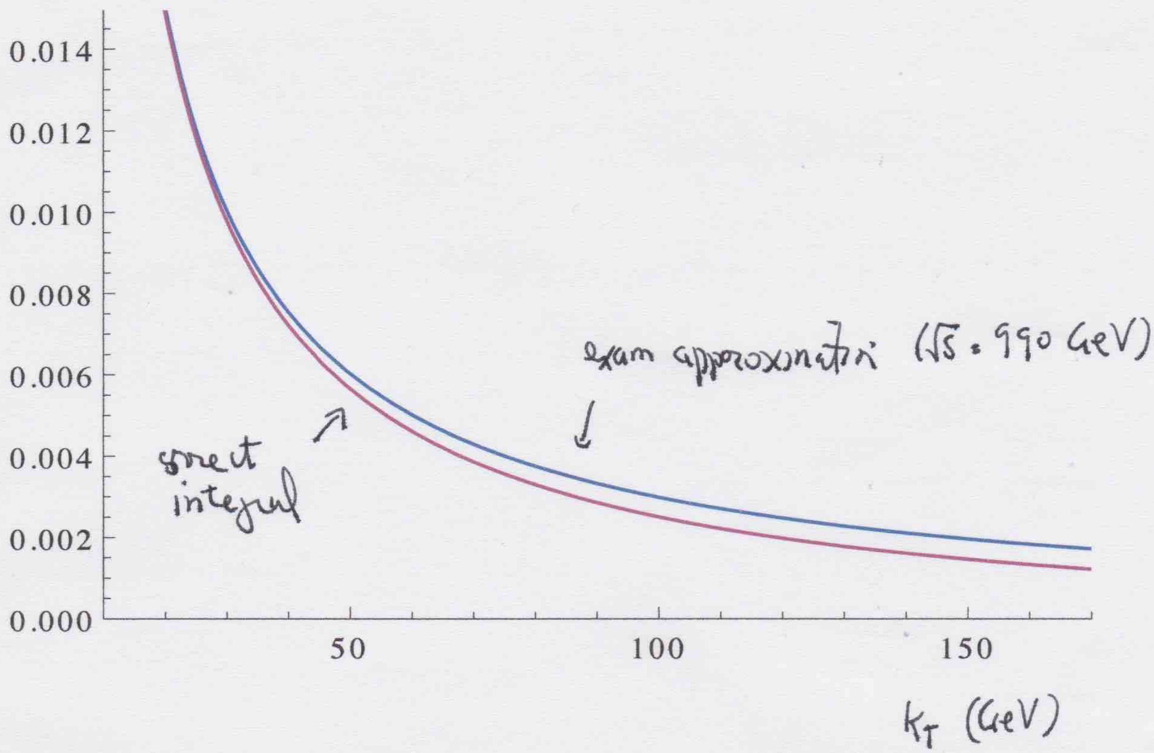
$$\approx \frac{\alpha_s^2}{72\pi} Y^2 \log\frac{m_H^2}{m_b^2}$$

$$\left[\frac{1}{s^2} \frac{k_T}{k_{\parallel}} \left\{ \frac{1}{k+k_{\parallel}}\left((k-k_{\parallel})^2 + \frac{m_H^4}{s}\right) + \frac{1}{k-k_{\parallel}}\left((k+k_{\parallel})^2 + \frac{m_H^4}{s}\right)\right\}\right]_{s = \frac{1}{2} s(gg)}$$

Here is a graph of this function at $s = (990\text{ GeV})^2$, and a comparison to the result of doing the full integral numerically:

$$\frac{d\sigma}{dk_T}(gg \to H + b\text{ jet})\Big|_{\sqrt{s}(gg) = 1400\text{ GeV}}$$

(pb)



j.) $\displaystyle\int_{30} dk_T \frac{d\sigma}{dk_T} \approx 0.6$ pb

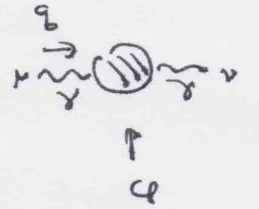
Then insisting on $k_T > 30$ GeV captures about 40% of the total cross section.

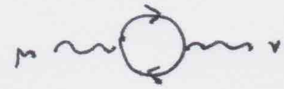
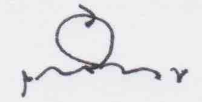
2.) a.)

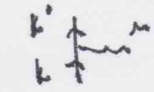
$$v = \frac{1}{(\sqrt{2}\,G_F)^{1/2}}$$

$$g = [4\sqrt{2}\,G_F m_W^2]^{1/2}$$

$$s_w = \frac{e}{g} = \left[\frac{\pi\alpha}{\sqrt{2}\,G_F m_W^2}\right]^{1/2}$$

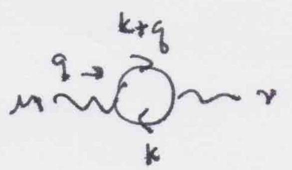
b.)  $= i\,\Pi^{\mu\nu}_{EM}(q)$

$=$  $+$ 

 $= i(k+k')^\mu eQ$  $= 2i g^{\mu\nu} e^2 Q^2$

Where $Q = I^3 + Y$ on elements of φ

$$\begin{pmatrix} \varphi \\ \vdots \\ \varphi \end{pmatrix} \begin{matrix} \leftarrow Q = I + Y \\ \\ \leftarrow Q = -I + Y \end{matrix}$$

 $= \operatorname{tr}(ieQ)^2 \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{i(2k+q)^\mu i(2k+q)^\nu}{(k^2-M^2)((k+q)^2-M^2)}$

$$= e^2 \operatorname{tr} Q^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2\ell+(1-2x)q)^\mu(2\ell+(1-2x)q)^\nu}{[\ell^2 + x(1-x)q^2 - M^2]^2}$$

$\ell = k + xq$

$k = \ell - xq$

$2k+q = 2\ell + (1-2x)q$

$$= e^2 \operatorname{tr} Q^2 \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \left\{ \frac{\Gamma(1-d/2)(-\frac{d}{2} g^{\mu\nu})}{\Delta^{1-d/2}} + (1-2x)^2 q^\mu q^\nu \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \right\}$$

$\Delta = M^2 - x(1-x)q^2$

$$= 2i\, g^{\mu\nu} e^2 \operatorname{tr} Q^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-M^2} \cdot \frac{(k+q)-M^2}{(k+q)^2-M^2}$$

$$= -2(\operatorname{tr} Q^2)\, e^2\, g^{\mu\nu} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell+(1-x)q)^2-M^2}{[\ell^2-\Delta]^2}$$

$$= -2e^2 \operatorname{tr} Q^2\, g^{\mu\nu} \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \Big\{ -\frac{d}{2} \frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + \left((1-x)^2 q^2 - M^2\right) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \Big\}$$

 + 

$$= e^2 \operatorname{tr} Q^2 \frac{i}{(4\pi)^{d/2}} \int_0^1 dx$$

$$\Big\{ g^{\mu\nu}(-2+d) \frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[q^\mu q^\nu (1-2x)^2 - 2(1-x)^2 q^2 g^{\mu\nu} + 2M^2 g^{\mu\nu} \right] \Big\}$$

$$= e^2 \operatorname{tr} Q^2 \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\Big\{ -2g^{\mu\nu}(\cancel{M^2} - x(1-x)q^2) + q^\mu q^\nu (1-2x)^2 - \overbrace{(x^2+(1-x)^2)}^{\text{symmetrize}} q^2 g^{\mu\nu} + \cancel{2M^2 g^{\mu\nu}} \Big\}$$

$$= e^2 \operatorname{tr} Q^2 \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \left[-(1-2x)^2\right] (g^{\mu\nu} q^2 - q^\mu q^\nu) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$= e^2 (\operatorname{tr} Q^2) \left(\frac{-i}{(4\pi)^2}\right) \int_0^1 dx\, (1-2x)^2$$

$$\cdot \left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\left(M^2 - x(1-x)q^2\right) \right] (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

so

$$\Pi_{EM}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \left(-\frac{e^2 \operatorname{tr} Q^2}{(4\pi)^2} \right)$$

$$\int_0^1 dx\, (1-2x)^2 \left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\left[M^2 - x(1-x) q^2\right] \right]$$

c) $$\Pi^{\mu\nu}_{EM}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{e^2\,\mathrm{tr}\,Q^2}{(4\pi)^2}\right)$$
$$\left\{ \frac{1}{3}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right] + \frac{1}{30}\frac{q^2}{M^2} + \cdots \right\}$$

d.) The measurable value of α is given by the strength of the Coulomb potential at $q^2=0$:

$$V(q) = \frac{\alpha}{q^2} = \frac{e^2}{4\pi}\frac{1}{q^2}\left(1 + \left(\Pi^{00}_{EM}(q^2)/q^2\right)\Big|_{q^2=0}\right)$$

$$\frac{\Delta\alpha}{\alpha} = \frac{\Pi^{00}}{q^2} = -\frac{e^2}{(4\pi)^2}\mathrm{tr}\,Q^2\left\{\frac{1}{3}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right] + \cdots\right\}$$

then

$$\frac{\Delta s_w}{s_w} = -\frac{e^2}{(4\pi)^2}\mathrm{tr}\,Q^2\,\frac{1}{6}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right]$$

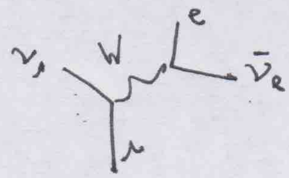
Actually, $\Delta g = 0$ at this level.

e.) Since all of the φ masses are equal, these vacuum polarization diagrams can be obtained from the EM vacuum polarization diagram by changing the gauge charges.

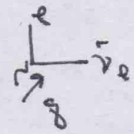
$$i\Pi^{\mu\nu}_{ZZ} = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{g^2}{c_w^2}\frac{1}{(4\pi)^2}\mathrm{tr}\,Q_Z^2\right)$$
$$\cdot\left\{\frac{1}{3}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right] + \frac{1}{30}\frac{q^2}{M^2}\right\}$$

$$i\Pi^{\mu\nu}_{WW} = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{g^2}{2}\frac{1}{(4\pi)^2}\mathrm{tr}(T^+T^-)\right)$$
$$\left\{\frac{1}{3}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right] + \frac{1}{30}\frac{q^2}{M^2}\right\}$$

f.) G_F is measured from μ^- decay.



q^μ on the lepton lines gives zero if we ignore the mass of the electron

q^μ  $= 0$

so we can write

$$i\Pi^{\mu\nu}_{WW} = i\Pi_{WW}(q)\, g^{\mu\nu}$$

The W propagator is modified by

$$g^{\mu\nu}\left[\frac{-i}{q^2 - \frac{g^2 v^2}{4}} + \frac{-i}{q^2 - \frac{g^2 v^2}{4}}\, i\Pi_{WW}(q)\, \frac{-i}{q^2 - g^2 v^2/4} + \cdots\right]$$

$$= \frac{-ig^{\mu\nu}}{\left[q^2 - g^2 v^2/4 - \Pi_{WW}(q)\right]}$$

G_F is evaluated at $q^2 = 0$, and $\Pi_{WW}(q) = 0$ there, so $\Delta G_F = 0$. We can also read off

$$\Delta m_W^2 = \Pi_{WW}(m_W^2) = m_W^2 \left[-\frac{g^2}{2}\frac{1}{(4\pi)^2}\, \mathrm{tr}(T^+T^-)\right]$$

$$\cdot \left\{ \frac{1}{3}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right] + \frac{1}{30}\frac{m_W^2}{M^2} \right\}$$

then.

with $T_1 = \left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2\right]$

g:) $$\frac{\Delta v}{v} = 0$$

$$\frac{\Delta g}{g} = \frac{1}{2}\frac{\Delta m_W^2}{m_W^2} = -\frac{g^2}{4}\frac{1}{(4\pi)^2}\,\mathrm{tr}(T^+T^-)\left(\frac{1}{3}L + \frac{1}{30}\frac{m_W^2}{M^2}\right)$$

$$\frac{\Delta s_w}{s_w} = \frac{1}{2}\frac{\Delta\alpha}{\alpha} - \frac{1}{2}\frac{\Delta m_W^2}{m_W^2}$$

$$= -\frac{1}{2}\frac{e^2}{(4\pi)^2}\,\mathrm{tr}\,Q^2\,\frac{1}{3}L + \frac{1}{4}\frac{g^2}{(4\pi)^2}\,\mathrm{tr}(T^+T^-)\left(\frac{1}{3}L + \frac{1}{30}\frac{m_W^2}{M^2}\right)$$

h.) Similarly to m_W^2 above

$$\frac{\Delta m_Z^2}{m_Z^2} = -\frac{g^2}{c_w^2}\frac{1}{(4\pi)^2}\,\mathrm{tr}\,Q_Z^2\left(\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right)$$

At leading order,

$$m_Z^2 = m_W^2\frac{1}{\cos^2\theta_w} = m_W^2\left(\frac{1}{1-s_w^2}\right)$$

So define

$$(m_Z^2)_0 \equiv m_W^2\left(1 - \frac{\pi\alpha}{\sqrt{2}G_F m_W^2}\right)^{-1}$$

At 1-loop order, we will find that $m_Z^2-(m_Z^2)_0$ is nonzero, and we will get a prediction for this quantity.

$$\frac{m_Z^2-(m_Z^2)_0}{m_Z^2} = \frac{\Delta m_Z^2}{m_Z^2} - \frac{\Delta m_W^2}{m_W^2} - \frac{m_W^2}{m_Z^2}\frac{\Delta s_w^2}{c_w^4}$$

$$= \frac{\Delta m_Z^2}{m_Z^2} - \frac{\Delta m_W^2}{m_W^2} - \frac{s_w^2}{c_w^2}\left(\frac{\Delta\alpha}{\alpha} - \frac{\Delta m_W^2}{m_W^2}\right)$$

$$= \frac{\Delta m_Z^2}{m_Z^2} - \frac{\Delta m_W^2}{m_W^2}\left(1 - \frac{s_w^2}{c_w^2}\right) - \frac{s_w^2}{c_w^2}\frac{\Delta\alpha}{\alpha}$$

Now evaluate the contributions to this from φ loop corrections

$$\frac{m_Z^2-(m_Z^2)_0}{m_Z^2} = -\frac{g^2}{c_W^2}\frac{1}{(4\pi)^2}\,\mathrm{tr}\,Q_Z^2\left(\tfrac{1}{3}L+\tfrac{1}{30}\frac{m_Z^2}{M^2}\right)$$

$$+\frac{g^2}{2}\frac{c_W^2-s_W^2}{c_W^2}\frac{1}{(4\pi)^2}\,\mathrm{tr}\,(T^+T^-)\left(\tfrac{1}{3}L+\tfrac{1}{30}\frac{m_W^2}{M^2}\right)$$

$e^2 = s_W^2 g^2$

$$+\frac{s_W^2}{c_W^2}\frac{s_W^2 g^2}{(4\pi)^2}\,\mathrm{tr}\,Q^2\ \tfrac{1}{3}L$$

First, work out the terms proportional to $L = \left(\frac{1}{\epsilon}+\cdots-\log M^2\right)$. We need

$$\mathrm{tr}\,Q^2 = \mathrm{tr}\,(T^3+Y)^2 = \mathrm{tr}\,(T^3)^2+\mathrm{tr}\,Y^2$$

$$\mathrm{tr}\,T^+T^- = \mathrm{tr}\,(T^1+iT^2)(T^1-iT^2) = \mathrm{tr}\left[(T^1)^2+(T^2)^2\right] = 2\,\mathrm{tr}(T^3)^2$$

$$\mathrm{tr}\,Q_Z^2 = \mathrm{tr}\left[T^3-s_W^2(T^3+Y)\right]^2 = \mathrm{tr}\,(c_W^2T^3-s_W^2Y)^2$$

$$= c_W^4\,\mathrm{tr}(T^3)^2+s_W^4\,\mathrm{tr}\,Y^2$$

since $\mathrm{tr}\,T^3=0$ over the φ multiplet. Then the L terms are

$$-\frac{g^2}{(4\pi)^2}\,\mathrm{tr}\,(T^3)^2\ \tfrac{1}{3}L$$

$$\cdot\left(c_W^2-\frac{c_W^2-s_W^2}{c_W^2}-\frac{s_W^4}{c_W^2}\right) \leftarrow (A)$$

$$-\frac{g^2}{(4\pi)^2}\,\mathrm{tr}\,Y^2\ \tfrac{1}{3}L$$

$$\cdot\left(\frac{s_W^4}{c_W^2}-\frac{s_W^4}{c_W^2}\right) \leftarrow 0\ !$$

$$(A) = c_W^2-1+\frac{s_W^2-s_W^4}{c_W^2} = c_W^2-1+\frac{s_W^2\cancel{c_W^2}}{\cancel{c_W^2}} = 0\ !$$

i) so all terms in I_1 cancel. This includes a cancellation of all $\log M^2$ terms.

j) Finally, work out the terms of order $\frac{1}{M^2}$:

$$\frac{m_Z^2 - (m_Z^2)_0}{m_Z^2} = -\frac{1}{30}\frac{g^2}{(4\pi)^2}\frac{1}{c_W^2}\left(c_W^4 \, tr(T^3)^2 + s_W^4 \, tr Y^2\right)\frac{m_Z^2}{M^2}$$

$$+ \frac{1}{30}\frac{g^2}{(4\pi)^2} \, tr(T^3)^2 \, \frac{m_W^2}{M^2}$$

$$= -\frac{1}{30}\frac{g^2}{(4\pi)^2}\left[tr(T^3)^2 \cdot \left(\frac{c_W^2 m_Z^2 - m_W^2}{M^2}\right) + \left(\frac{s_W^4}{c_W^2}\frac{m_Z^2}{M^2}\right) tr Y^2\right]$$

$$= -\frac{1}{30}\frac{g^2}{(4\pi)^2}\frac{s_W^4}{c_W^2}(tr Y^2)\frac{m_Z^2}{M^2}$$

the $tr(T^3)^2$ terms cancel; $tr Y^2 = Y^2(2I+1)$.

Finally,

$$\frac{[m_Z^2 - (m_Z^2)_0]}{m_Z^2} = -\frac{1}{30}\frac{g^2}{(4\pi)^2}\frac{s_W^2}{c_W^2} Y^2(2I+1)\frac{m_Z^2}{M^2}$$

or

$$m_Z - (m_Z)_0 = -\frac{1}{60}\frac{s_W^2}{c_W^2}\frac{g^2}{(4\pi)^2}(2I+1)Y^2 \frac{m_Z^3}{M^2}$$

$$= -\frac{10. \text{ GeV}^3}{M^2} \cdot (2I+1) Y^2$$

the bound suggested in the problem set is

$$0.05 \text{ GeV} > \frac{10. (\text{GeV})^3}{M^2} (2I+1) Y^2$$

or

$$M^2 > (2I+1) Y^2 \cdot 200 \text{ GeV}^2$$

for $I=1, Y=1$

$$M > 25 \text{ GeV} \quad \text{(a rather weak bound)}$$

for $Y=0$ the bound disappears.

k) Putting $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

$$\Delta \mathcal{L} = -\gamma \frac{v^2}{4} \varphi^\dagger T^3 \varphi$$

This gives a spectrum of masses.

$$m_\varphi^2 = \left(M^2 + \gamma \frac{v^2}{2} T^3\right)$$

or

$$m_\varphi(T^3) \cong M + \frac{\gamma v^2}{4M} T^3 + \cdots$$

for $T^3 = -I, -(I-1) \cdots ; +I$

l.) For γ and Z, the diagrams all have particles of the same mass, so the resulting vacuum polarizations are basically unchanged:

$$\Pi_{EM}(q^2) = -\frac{e^2}{(4\pi)^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$\mathrm{tr}\, Q^2 \int_0^1 dx\, (1-2x)^2 \left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\left[M^2 + \frac{\gamma v^2}{2} T^3 - x(1-x) q^2\right]\right]$$

$$= -\frac{e^2}{(4\pi)^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$\mathrm{tr}\, Q^2 \left\{ \frac{1}{3} L + \left(-\frac{1}{3}\right) \frac{\frac{\gamma v^2}{2} T^3}{M^2} + \frac{1}{30}\frac{q^2}{M^2} \right\}$$

$Q = T^3 + Y$. Now $\mathrm{tr}\,(T^3)^3 = 0$ $\mathrm{tr}\, T^3 = 0$ so

$$\Pi_{EM}(q^2) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\left[-\frac{e^2}{(4\pi)^2}\right]$$

$$\left\{ \mathrm{tr}\, Q^2 \cdot \left(\frac{1}{3} L + \frac{1}{30}\frac{q^2}{M^2}\right) + \mathrm{tr}(T^3)Y \cdot \left(\frac{-1}{3}\frac{\gamma v^2}{M^2}\right)\right\}$$

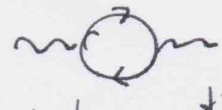
similarly

$$\Pi_{ZZ}(q^2) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\left[-\frac{g^2}{c_w^2}\frac{1}{(4\pi)^2}\right]$$

$$\cdot \left\{ \mathrm{tr}\, Q_Z^2 \left(\frac{1}{3} L + \frac{1}{30}\frac{q^2}{M^2}\right) + \mathrm{tr}\, Q_Z^2 T^3 \left(\frac{-1}{6}\frac{\gamma v^2}{M^2}\right)\right\}$$

$$\mathrm{tr}\, Q_Z^2 T^3 = \mathrm{tr}\left(c_w^2 T^3 - s_w^2 Y\right)^2 T^3 = 2 c_w^2 s_w^2 \left(\mathrm{tr}(T^3)^2\right) Y$$

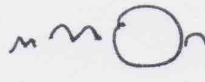
The W vacuum polarization requires more care, since the diagram



contains φ particles with 2 different masses.

Lets recompute the vacuum polarization with masses. $m_1\ m_2$

$$m_1^2 = m_0^2 + \tfrac{1}{2}\delta m^2 \qquad m_2^2 = m_0^2 - \tfrac{1}{2}\delta m^2 .$$



$$= \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{(2k+q)^\mu(2k+q)^\nu}{(k^2-m_1^2)((k+q)^2-m_2^2)}$$

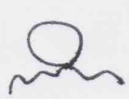
$$= \frac{g^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{(2\ell+(1-2x)q)^\mu(2\ell+(1-2x)q)^\nu}{[\ell^2-\Delta]^2}$$

where now
$$\Delta = x m_1^2 + (1-x)m_2^2 - x(1-x)q^2$$
$$= m_0^2 - \left(\tfrac{1-2x}{2}\right)\delta m^2 - x(1-x)q^2$$

$$= \frac{g^2}{2}\frac{i}{(4\pi)^{d/2}}\int_0^1 dx\left\{-\frac{4}{2}g^{\mu\nu}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}(1-2x)^2 q^\mu q^\nu\right\}$$

For the other diagram, we should remember that the vertex is actually

 $2i\{T_0^+,T^-\}\, g^{\mu\nu}\,\frac{g^2}{2}$ so

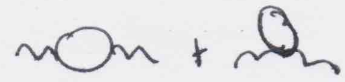


$$= -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\, g^{\mu\nu}\left(\frac{1}{(k^2-m_1^2)}+\frac{1}{(k+q)^2-m_2^2}\right)$$

$$= -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\, g^{\mu\nu}\,\frac{k^2-m_1^2+(k+q)^2-m_2^2}{(k^2-m_1^2)((k+q)^2-m_2^2)}$$

$$= -\frac{g^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\, g^{\mu\nu}\,\frac{(\ell-xq)^2+(\ell+(1-x)q)^2-m_1^2-m_2^2}{[\ell^2-\Delta]^2}$$

$$= -\frac{g^2}{2}\frac{i}{(4\pi)^{d/2}}\int_0^1 dx\left\{-2\frac{d}{2}g^{\mu\nu}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + g^{\mu\nu}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left(x^2+(1-x)^2-m_1^2-m_2^2\right)\right\}$$



$$= \frac{i}{(4\pi)^{d/2}} \frac{g^2}{2} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \Big\{ -2 g^{\mu\nu} \big(m_0^2 - (\tfrac{1-2x}{2}) \delta m^2 - x(1-x) q^2\big)$$

$$- g^{\mu\nu} \big((x^2 + (1-x)^2) q^2 - 2 m_0^2\big)$$

$$+ (1-2x)^2 q^\mu q^\nu \Big\}$$

$$= \frac{i}{(4\pi)^{d/2}} \frac{g^2}{2} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \Big[-(1-2x)^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) + 2 (\tfrac{1-2x}{2}) \delta m^2 g^{\mu\nu} \Big]$$

$$= \frac{-i}{(4\pi)^2} \frac{g^2}{2} \int_0^1 dx \, \Big\{ \frac{1}{\epsilon} - \gamma + \log 4\pi - \log \big[m_0^2 - (\tfrac{1-2x}{2}) \delta m^2 - x(1-x) q^2 \big] \Big\}$$

$$\cdot \Big((1-2x)^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) - (1-2x) \delta m^2 g^{\mu\nu} \Big)$$

$$= \frac{-i}{(4\pi)^2} \frac{g^2}{2} \int_0^1 dx \, \Big\{ \frac{1}{\epsilon} - \gamma + \log 4\pi - \log m_0^2 + \frac{(1-2x)\delta m^2}{2 m_0^2} + \frac{x(1-x)}{m_0^2} q^2 \Big\}$$

$$(1-2x)^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) - (1-2x) \delta m^2 g^{\mu\nu} \Big\}$$

Note that this vacuum polarization amplitude is no longer transverse, but that this problem comes into play only at order $(\delta m^2)^2$. The terms linear in δm^2 vanish.

Then also $\Pi^{\mu\nu}_{WW}(q^2) = \mathcal{O}((\delta m^2)^2)$ at $q^2 = 0$, which implies

$$\Delta G_F = 0 + \mathcal{O}((\delta m^2)^2) = 0 + \mathcal{O}(\gamma^2)$$

For the term that involves I^3 & I^3+1, $\log m_0^2 = \log M^2 + \frac{\gamma v^2}{2M^2}(I^3 + \tfrac{1}{2})$

To first order in γ, these extra terms add to 0 over the multiplet.

Then, the W vacuum polarization gives no extra term.

In all, we find

$$\frac{m_Z^2 - (m_Z^2)_0}{m_Z^2} = (\text{previous terms})$$

$$- \frac{g^2}{c_w^2} \frac{1}{(4\pi)^2} \left(-\frac{1}{3} s_w^2 c_w^2 \, tr (T^3)^2 Y \right) \frac{\gamma v^2}{M^2}$$

$$+ \frac{s_w^4}{c_w^2} \frac{g^2}{(4\pi)^2} \left(-\frac{1}{3} \, tr (T^3)^2 Y \right) \frac{\gamma v^2}{M^2}$$

$$= (\text{previous}) + \frac{1}{3} \frac{g^2}{(4\pi)^2} s_w^2 \left(1 - \frac{s_w^2}{c_w^2}\right) tr (T^3)^2 Y \frac{\gamma v^2}{M^2}$$

$$m_Z - (m_Z)_0 = (\text{previous result})$$

$$+ \frac{1}{6} \frac{g^2}{(4\pi)^2} s_w^2 \left(1 - \frac{s_w^2}{c_w^2}\right) tr (T^3)^2 Y \frac{\gamma v^2 m_Z}{M^2}$$