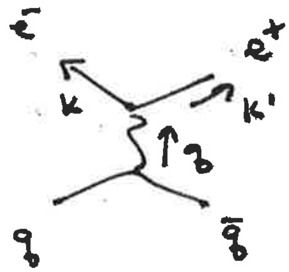
Physics 331 - Final Exam 2019

Solutions

1.) a,b) The decay amplitude is



$$i\mathcal{M} = (-ie)^2 \frac{-i}{q^2} Q \, \bar{u}(k)\gamma^\mu v(k') \, \bar{v}(\bar{q})\gamma_\mu u(q)$$

for the quarks (at rest)

$$u(q) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \qquad v(\bar{q}) = \sqrt{m}\begin{pmatrix}\bar{\xi} \\ -\bar{\xi}\end{pmatrix}$$

$$\gamma^0\gamma^\mu = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}$$

$$= \begin{pmatrix}\bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu\end{pmatrix}$$

so

$$\bar{v}\gamma^\mu u = m\,(\bar{\xi}^\dagger\bar{\sigma}^\mu\xi - \bar{\xi}^\dagger\sigma^\mu\xi)$$

$$= 2m\,\bar{\xi}^\dagger(0 \quad -\vec{\sigma})^\mu\,\xi$$

for the e^+e^- choose axes so that $e^- \parallel \hat{3}$

$e^-_L e^+_R$: $\quad u(k) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v(k) = \sqrt{2E}\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$

$$\bar{u}\gamma^\mu v = 2E\,(0\,1)\,\bar{\sigma}^\mu \begin{pmatrix}-1\\0\end{pmatrix}$$

$$= +2E\,(0, 1, i, 0)$$

$e^-_R e^+_L$: $\quad u(k) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v(k) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$\bar{u}\gamma^\mu v = 2E\,(1\,0)\,\sigma^\mu \begin{pmatrix}0\\1\end{pmatrix}$$

$$= 2E\,(0, 1, -i, 0)$$

so

$$i\mathcal{M}_{L,R} = i\,\frac{e^2}{q^2}\,Q \cdot 2m \cdot 2E\ \xi^\dagger (\sigma^1 \pm i\sigma^2)\,\xi$$

$$= i\,e^2 Q\ \xi^\dagger(\sigma^1 \pm i\sigma^2)\,\xi \qquad \begin{aligned} 2m &\cong M \\ 2E &\cong M \\ q^2 &= M^2 \end{aligned}$$

so I in eq. (2) on the problem set $\propto \vec{\sigma}$

$$\operatorname{tr} I = 0 \qquad \operatorname{tr} \vec{\sigma}^i I = 2ie^2 Q\,(\delta^{i1} \pm i\delta^{i2})$$

so the $S=0$ bound state does not decay to e^+e^-

the $S=1$ bound state can decay in this way

[Why? The decay through 1 γ requires $C = -1$.

the $S=0$ bound state has $C = +1$

the $S=1$ bound state has $C = -1$]

For the $S=1$ bound state

$$\mathcal{M}(B \to e^+e^-) = \sqrt{\frac{2}{M}} \overbrace{\int \frac{d^3k}{(2\pi)^3} \psi(k)}^{\psi(0)} \frac{2e^2 Q}{\sqrt{2}} (\epsilon^1 \pm i\epsilon^2)$$

$$\times \text{(color factor)}$$

The quark-antiquark bound state is a color singlet; the color wavefunct is $\frac{1}{\sqrt{3}} \delta^{ij}$

and the photon vertex has a δ^{ij}

so color factor $= \sqrt{3}$.

Finally, we need to sum over e^+e^- polarization states and integrate over their phase space.

$$\Gamma(B \to e^+e^-) = \frac{1}{2M} \int d\Pi_2 \; \frac{2}{M} |\psi(0)|^2 \, \frac{4e^4}{2} Q^2$$

$$\cdot \; 3 \cdot \left(|(\epsilon^1 + i\epsilon^2)|^2 + |\epsilon^1 - i\epsilon^2|^2 \right)$$

to integrate over the $\bar{e}$ direction, it is equivalent to average over the B polarization direction. Then

$$\Gamma(B \to e^+e^-) = \frac{1}{2M}\,\frac{2}{M}\,\frac{1}{8\pi}\,|\psi(0)|^2\;\frac{4e^4Q^2}{2}$$

$$\cdot\, 3 \,\cdot\, \left(\frac{2}{3} + \frac{2}{3}\right)$$

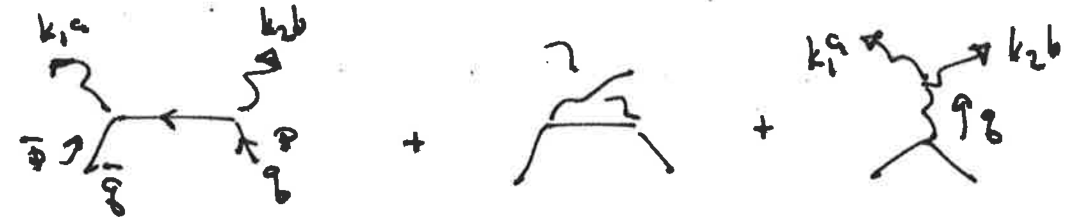
$$= \frac{e^4}{\pi}\,Q^2\,\frac{1}{M^2}\,|\psi(0)|^2$$

so

$$\Gamma(B \to e^+e^-) = \frac{16\pi\alpha^2}{M^2}\,Q^2\,|\psi(0)|^2$$

for $S=1$ only ; 0 for $S=0$

e.) for $B \to gg$ the decay matrix element is

$\mathcal{M} =$ 

$$= \bar{v}(\bar{q})\Big\{ (+ig\gamma\cdot\varepsilon^*(k_1))\,\frac{i(\not p - \not k_2)+m}{(p-k_2)^2-m^2}\,(+ig\gamma\cdot\varepsilon^*(k_1))\; t^a t^b$$

$$+ \;(+ig\gamma\cdot\varepsilon^*(k_2))\,\frac{i(\not p - \not k_1)+m}{(p-k_1)^2-m^2}\,(+ig\gamma\cdot\varepsilon^*(k_2))\; t^b t^a$$

$$+ \; -g f^{abc}\left[\varepsilon_1^*\cdot\varepsilon_2^*\,(k_1-k_2)^\lambda + \varepsilon_2^{*\lambda}(k_2+q)\cdot\varepsilon_1^* + \varepsilon_1^{*\lambda}(-q-k_1)\cdot\varepsilon_2^*\right]$$

$$\cdot\,(+ig)\,\gamma_\lambda t^c\;\frac{-i}{q^2}\;\Big\}\,u(q)$$

Let's now evaluate this for quarks at rest and for gluons of definite helicity

$$p = (m, \vec{0}) \qquad \bar{p} = (m, \vec{0})$$

$$k_1 = (m, 0, 0, m) \qquad k_2 = (m, 0, 0, -m) \qquad \vec{k}_1 \parallel \hat{3}$$

$$\varepsilon_1 = \begin{cases} \frac{1}{\sqrt{2}}(0\ 1\ i\ 0) & R \\ \frac{1}{\sqrt{2}}(0\ 1\ -i\ 0) & L \end{cases} \qquad \varepsilon_2 = \begin{cases} \frac{1}{\sqrt{2}}(0\ -1\ i\ 0) & R \\ \frac{1}{\sqrt{2}}(0\ -1\ -i\ 0) & L \end{cases}$$

$$(p-k_1)^2 - m^2 = (p-k_2)^2 - m^2 = -2m^2$$

$$\varepsilon_1^* \cdot \gamma = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\sigma^1 + i\sigma^2 \\ \sigma^1 - i\sigma^2 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & -\sigma^- \\ \sigma^- & 0 \end{pmatrix} & R \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\sigma^1 - i\sigma^2 \\ \sigma^1 + i\sigma^2 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & -\sigma^+ \\ \sigma^+ & 0 \end{pmatrix} & L \end{cases}$$

$$\varepsilon_2^* \cdot \gamma = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma^1 + i\sigma^2 \\ -\sigma^1 - i\sigma^2 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & \sigma^+ \\ -\sigma^+ & 0 \end{pmatrix} & R \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma^1 - i\sigma^2 \\ -\sigma^1 + i\sigma^2 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & \sigma^- \\ -\sigma^- & 0 \end{pmatrix} & L \end{cases}$$

Since $\sigma^+\sigma^+ = \sigma^-\sigma^-$, it will turn out that

$$\mathcal{M}(B \to g_1(L)\, g_2(R)) = \mathcal{M}(B \to g_1(R)\, g_2(L)) = 0$$

This is actually a consequence of angular momentum conservation.

The state $g_1(L)\, g_2(R)$ is

so this is a state with $J^3 = -2$. Since

B has either spin 0 or spin 1, $B \to g_1(L)\, g_2(R)$ is forbidden. This applies also to $B \to g_1(R)\, g_2(L)$.

Now,

$$+ig\, \varepsilon_1^* \cdot \gamma \; \frac{i\,(\not p - \not k_2) + m}{(p-k_2)^2 - m^2} \; +ig\,\varepsilon_2^* \cdot \gamma$$

for RR

$$= -ig^2 \frac{1}{(-2m^2)} (\sqrt{2})^2 \begin{pmatrix} 0 & -\sigma^- \\ \sigma^- & 0 \end{pmatrix} \begin{pmatrix} m & -m\sigma^3 \\ m\sigma^3 & m \end{pmatrix} \begin{pmatrix} 0 & \sigma^+ \\ -\sigma^+ & 0 \end{pmatrix}$$

$$= ig^2 \frac{1}{m} \begin{pmatrix} -\sigma^-\sigma^3 & -\sigma^- \cdot 1 \\ \sigma^- \cdot 1 & -\sigma^-\sigma^3 \end{pmatrix} \begin{pmatrix} 0 & \sigma^+ \\ -\sigma^+ & 0 \end{pmatrix} \quad \leftarrow \sigma^-\sigma^3 = +\sigma^-$$

$$= ig^2 \frac{1}{m} \begin{pmatrix} \sigma^-\sigma^+ & -\sigma^-\sigma^+ \\ \sigma^-\sigma^+ & \sigma^-\sigma^+ \end{pmatrix}$$

$$\bar{v} \text{ (above) } u = \sqrt{m}\,(-\xi^\dagger \;\; +\xi^\dagger) \; [\text{above}] \; \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= ig^2 (+2\, \xi^\dagger\, \sigma^-\sigma^+\, \xi)$$

$$= +2ig^2 (\xi^\dagger\, \sigma^-\sigma^+\, \xi)$$

similarly, for LL

$$= +ig^2 \frac{m}{m} \begin{pmatrix} -\sigma^+\sigma^3 & -\sigma^+ \\ \sigma^+ & -\sigma^+\sigma^3 \end{pmatrix} \begin{pmatrix} 0 & \sigma^- \\ -\sigma^- & 0 \end{pmatrix} \qquad \sigma^+\sigma^3 = -\sigma^+$$

$$= +ig^2 \frac{m}{m} \begin{pmatrix} \sigma^+\sigma^- & \sigma^+\sigma^- \\ -\sigma^+\sigma^- & \sigma^+\sigma^- \end{pmatrix}$$

and

$$\bar{v}^* [\text{above}]\, u = -2ig^2\, \xi^\dagger \sigma^+\sigma^- \xi$$

next

$$+ig\ \varepsilon_2^*\cdot\gamma\ \frac{i(\not p-\not k_1)+m}{(p-k_1)^2-m^2}\ +ig\,\varepsilon_1^*\cdot\gamma$$

for RR

$$= ig^2\,\frac{1}{-2m^2}(\sqrt{2})^2\begin{pmatrix}0 & \sigma^+\\ -\sigma^+ & 0\end{pmatrix}\begin{pmatrix}m & +m\sigma^3\\ -m\sigma^3 & m\end{pmatrix}\begin{pmatrix}0 & -\sigma^-\\ \sigma^- & 0\end{pmatrix}$$

$$= ig^2\,\frac{m}{m}\begin{pmatrix}-\sigma^+\sigma^3 & \sigma^+\\ -\sigma^+ & -\sigma^+\sigma^3\end{pmatrix}\begin{pmatrix}0 & -\sigma^-\\ \sigma^- & 0\end{pmatrix} \qquad \sigma^+\sigma^3 = -\sigma^+$$

$$= ig^2\,\frac{m}{m}\begin{pmatrix}\sigma^+\sigma^- & -\sigma^+\sigma^-\\ \sigma^+\sigma^- & \sigma^+\sigma^-\end{pmatrix}$$

$$\bar v(\cdots)u = 2ig^2\,\xi^\dagger\sigma^+\sigma^-\xi$$

for LL

$$= ig^2\,\frac{m}{m}\begin{pmatrix}-\sigma^-\sigma^3 & \sigma^-\\ -\sigma^- & -\sigma^-\sigma^3\end{pmatrix}\begin{pmatrix}0 & -\sigma^+\\ \sigma^+ & 0\end{pmatrix} \qquad \sigma^-\sigma^3 = +\sigma^-$$

$$= ig^2\begin{pmatrix}\sigma^-\sigma^+ & \sigma^-\sigma^+\\ -\sigma^-\sigma^+ & \sigma^-\sigma^+\end{pmatrix}$$

$$\bar v(\cdot\cdot)u = -2ig^2\,\xi^\dagger\sigma^-\sigma^+\xi$$

so the first two diagrams give

$$i\mathcal{M} = \xi^\dagger\left[+2ig^2\right]\left[\sigma^-\sigma^+\,t^at^b + \sigma^+\sigma^-\,t^bt^a\right]\xi$$

for RR

$$i\mathcal{M} = \xi^\dagger(2ig^2)\left[\sigma^+\sigma^-\,t^at^b + \sigma^-\sigma^+\,t^bt^a\right]\xi$$

for LL

the third diagram is

$$(+ig)(-gf^{abc}t^c)\,\frac{-i}{(4m^2)}\;\bar v\,\gamma^\lambda u$$

$$\cdot\left\{\epsilon_1^*\cdot\epsilon_2^*\,(0\;0\;0\;2m)^\lambda + \epsilon_2^{*\lambda}\,(2m,0\;0\;-m)\cdot\epsilon_1^* - \epsilon_1^{*\lambda}\,(2m,0,0,m)\cdot\epsilon_2^*\right\}$$

Now, the ϵ's are transverse, so

$$(2m,\;0\;0\;\pm m)\cdot\epsilon^*_{1,2} = 0$$

$$\epsilon_1^*\cdot\epsilon_2^* = +1 \quad \text{for RR, LL}, \quad = 0 \text{ for RL, LR}$$

$$(0\;0\;0\;2m)^\lambda\,\gamma^\lambda = 2m\begin{pmatrix}0 & -\sigma^3\\ \sigma^3 & 0\end{pmatrix}$$

also $\quad if^{abc}t^c = [t^a,t^b]$

so we find

$$= +ig^2\,[t^a,t^b]\,\frac{1}{2m}\,\bar v\begin{pmatrix}0 & -\sigma^3\\ \sigma^3 & 0\end{pmatrix}u$$

$$= +ig^2\,[t^a,t^b]\;\xi^\dagger\sigma^3\xi$$

$$\sigma^-\sigma^+ = \tfrac{1}{2}(1-\sigma^3) \qquad \sigma^+\sigma^- = \tfrac{1}{2}(1+\sigma^3)$$

in all

 $= \xi^\dagger\,(+ig^2)\left(\{t^a,t^b\} - \sigma^3[t^a,t^b] + \sigma^3[t^a,t^b]\right)\xi$

$$= \xi^\dagger\left[ig^2\,\{t^a,t^b\}\cdot 1\right]\xi \quad \text{for RR}$$

$$= \xi^\dagger\,(-ig^2)\left(\{t^a t^b\} + \sigma^3[t^a,t^b] - \sigma^3[t^a,t^b]\right)\xi$$

again

$$\left\{\begin{aligned} &= \xi^\dagger\,(ig^2)\cdot\tfrac{1}{2}\cdot\{t^a,t^b\}\,\xi \qquad RR \\ &= \xi^\dagger\,(-ig^2)\,\tfrac{1}{2}\,\{t^a,t^b\}\,\xi \qquad LL \end{aligned}\right.$$

the decay matrix elements are

$$\mathcal{M}(B\to gg) = \sqrt{\frac{2}{M}}\int\frac{d^3k}{(2\pi)^3}\,\psi(k)\,(\pm ig^2)\,\mathrm{tr}\left(\frac{1}{\sqrt{2}}\right)$$

$$\cdot\,(\text{color})$$

color is a trace of $\{t^a,t^b\}$ with the color wavefunction

$$\frac{1}{\sqrt{3}}\,\mathbb{1} \qquad\qquad (\mathrm{tr}\,t^a t^b = \tfrac{1}{2}\delta^{ab})$$

so

$$\text{color} = \frac{1}{\sqrt{3}}\,\mathrm{tr}\{t^a,t^b\} = \frac{1}{\sqrt{3}}\,\delta^{ab}$$

$$\mathcal{M}(B\to gg) = \sqrt{\frac{2}{M}}\,\psi(0)\,(\pm ig^2)\,\sqrt{2}\,\frac{1}{\sqrt{3}}\,\delta^{ab}$$

$$\sum_{\substack{\text{gluon}\\ \text{spin+color}}}|\mathcal{M}|^2 = \frac{2}{M}\,|\psi(0)|^2\,g^4\,\frac{2}{3}\cdot 2\cdot 8$$

$$= \frac{64}{3M}\,g^4\,|\psi(0)|^2$$

$$\Gamma(B\to gg) = \frac{1}{2M}\,\frac{1}{8\pi}\,\frac{64}{3M}\,g^4\,|\psi(0)|^2\left(\frac{1}{2}\right)$$

identical gluons

$$\Gamma(B \to gg) = \begin{cases} \dfrac{32\pi\,\alpha_s^2}{3} \, \dfrac{|\psi(0)|^2}{M^2} & S=0 \\ 0 & S=1 \end{cases}$$

This is similar to positronium

the $S=0$ state is $C=+1$, decays to $2\,\gamma$'s

$S=1$ state is $C=-1$, cannot decay to $2\,\gamma$'s

d.) Represent nonrelativistic sources by Wilson lines,

the expectation value of 2 sources is

$$= 1 + ig\int_{-\infty}^{\infty} dt_A \; ig \int_{-\infty}^{\infty} dt_B \; T_A^a \langle A^{0a}(\vec{x}) A^{0b}(0) \rangle T_B^b$$

$$= 1 + (ig)^2 \int dt_A \int dt_B \; T_A^a T_B^a \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2} e^{+i\vec{q}\cdot\vec{x} - iq^0(t^a - t^b)}$$

$$= 1 + (-g^2) \int dt_A \; T_A^a T_B^a \int \frac{d^4q}{(2\pi)^4} \frac{-i}{\underbrace{q^2}_{(q^0)^2 - |\vec{q}|^2}} 2\pi\delta(q^0) \, e^{i\vec{q}\cdot\vec{x}}$$

$$= 1 - i\, g^2 \, T_A^a T_B^a \int dt_A \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{q}|^2} + \cdots$$

so

$$E_{\text{energy}} = g^2 T_A^a T_B^a \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{q}|^2}$$

or

$$V(q) = \frac{g^2}{|\vec{q}|^2} T_A^a T_B^a$$

The object $T_A^a T_B^a$ is evaluated by

$$T_A^a T_B^a = \frac{1}{2}\left((T_A + T_B)^2 - T_A^2 - T_B^2\right)$$

$$= \frac{1}{2}\left(C_2(A \times B) - C_2(A) - C_2(B)\right)$$

for the 3 $\quad C_2(3) = C_2(\bar{3}) = \frac{4}{3}$ $\quad$ for SU(3)

for the 8 $\quad C_2(8) = C_2(G) = 3$

then

$$3 \times 3 \to 1 \qquad T_A^a T_B^a = -\frac{4}{3}$$

$$8 \times 8 \to 1 \qquad T_A^a T_B^a = -3$$

(in both cases the potential is attractive)

e) For $8 \times 8 \to 8$ (any 8)

$$T_A^a T_B^a = -\frac{3}{2}$$ (still attractive)

f.) For an $8\times 8 \to 1$ bound state, the rate calculation is the same as in (c) except that there is a new color factor. In (c) we had

$$\sum_{\text{colors}} \left(\frac{1}{\sqrt{3}} \delta_{ij} \{t^a, t^b\}_{ij} \right)^2 = \sum_{\text{colors}} \frac{1}{3} \left(tr[t_3^a t_3^b] \cdot 2 \right)^2$$

$$= \sum_{ab} \frac{1}{3} \cdot 4 \cdot \frac{1}{2}\delta^{ab} \frac{1}{2}\delta^{ab} = \frac{8}{3}$$

Here we have

$$\sum_{ab} \left(\frac{1}{\sqrt{8}} \delta_{mn} \{t_8^a, t_8^b\}_{mn} \right)^2 = \sum_{ab} \frac{1}{8} \cdot 4 \cdot \left((t_8^a t_8^b)_{mm} \right)^2$$

In the 8 = adjoint rep $\quad (t^a)_{mp} = i f^{map}$

so the above has

$$[t_8^a, t_8^b]_{mm} = -f^{map} f^{pbm} = f^{pma} f^{pmb} = 3\delta^{ab}$$

Then the color factor is

$$\frac{1}{8} \cdot 4 \cdot (3)^2 \cdot 8 = 36$$

Replacing $\quad \frac{8}{3} \to 36 \quad$ on p. 9 we find

$$\Gamma(B(8\times 8 \to 1, S=0) \to gg) = 144\pi\alpha_s^2 \frac{|\psi(0)|^2}{M}$$

There is another way to compute this

$$(t_8^a t_8^b)_{mn} = (i)^2 f^{map} f^{pbn}$$

$$= f^{apm} f^{bpn}$$

$$= -(t_8^p)_{am} (t_8^p)_{pn}$$

acting on any 8×8 wavefunction Ψ_{mn}. We can compute this by

$$(t^a t^b)_{mn} \Psi_{mn} = -(t_8^p)_{am}(t^p)_{bn}\Psi_{mn}$$

$$= -\frac{1}{2}\left(C_2(R) - C_2(8) - C_2(8)\right)\Psi_{ab}$$

where R is the composite representation.

For $8\times 8 \to 1$ $\quad R = 1$ and the above

$$= -\frac{1}{2}(0 - 3 - 3)\Psi_{ab} = +3\,\Psi_{ab}$$

Ψ is normalized, so $\sum_{ab} \Psi^*_{ab}\Psi_{ab} = 1$ and we get

$$\sum_{ab} \left|(\{t^a, t^b\})_{mn}\Psi_{mn}\right|^2 = (2)^2\cdot(3)^2\cdot 1 = 36$$

as above. This trick solves the next two examples.

(g) $\quad 8\times 8 \to 8_{\text{sym}} \qquad (\{t^a, t^b\})_{mn}\Psi_{mn} = 2\cdot\frac{3}{2}\cdot\Psi_{mn}$

so

$$\Gamma(B(s=0)\to 2g) = 72\pi\alpha_s^2 \frac{|\Psi(0)|^2}{M}$$

(h) $\quad 8\times 8 \to 8_{\text{antisym}}$

again

$$\Gamma(B(s=0)\to 2g) = 72\pi\alpha_s^2 \frac{|\Psi(0)|^2}{M}$$

2.) a.) The general form of an $SU(2)\times U(1)$ covariant derivative is

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} W_\mu^+ I^+ - i\frac{g}{\sqrt{2}} W_\mu^- I^-$$

$$-ieA_\mu Q \quad - i\frac{g}{c_w} Z_\mu (I^3 - Q s_w^2)$$

where $Q = I^3 + Y$ $\qquad c_w = \cos\theta_w \quad s_w = \sin\theta_w$

then for $I = \frac{1}{2}$ $Y = \frac{1}{2}$ $\qquad Q = \begin{cases} +1 & \text{for } E^+ \\ 0 & \text{for } E^0 \end{cases}$

The transformation laws of E_L, E_R and $\mathcal{E}$ are

$$E_L \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2 + i\beta\frac{1}{2}} E_L$$

$$E_R \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2 + i\beta\frac{1}{2}} E_R$$

$$\mathcal{E} \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2 + i\beta\frac{1}{2}} \mathcal{E}$$

all with the <u>same</u> transformation law

so that the mass terms

$$\delta\mathcal{L} = -M(\bar{E}_L E_R + \bar{E}_R E_L)$$

$$\delta\mathcal{L} = -M^2|\mathcal{E}|^2$$

are $SU(2)\times U(1)$ invariant.

b.) Since E_L and E_R have the same interactions with the W, A, Z, the Lagrangian of $E_L\, E_R$ can be written in terms of

$$E = \begin{pmatrix} E_L \\ E_R \end{pmatrix}$$

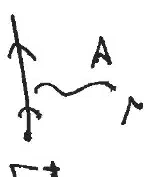
For example $\quad \left(\partial_\mu - i\frac{g}{\sqrt{2}} W^+ I^+\right) E$

$$= \partial_\mu E - i\frac{g}{\sqrt{2}} W^+_\mu I^+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} E - i\frac{g}{\sqrt{2}} W^+_\mu I^+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} E$$

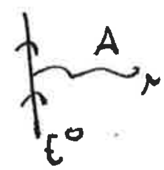
$$= \left(\partial_\mu - i\frac{g}{\sqrt{2}} W^+ I^+\right) E$$

so no factor of $\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ appear.

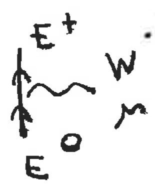
The Feynman rules are



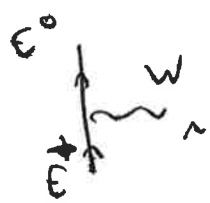
$= ie\gamma^\mu$



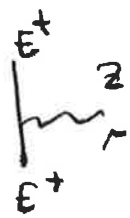
$= 0 \qquad e = s_W g$



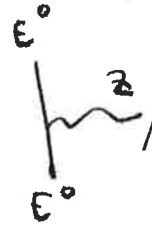
$= \frac{ig}{\sqrt{2}}\gamma^\mu$



$= \frac{ig}{\sqrt{2}}\gamma^\mu$

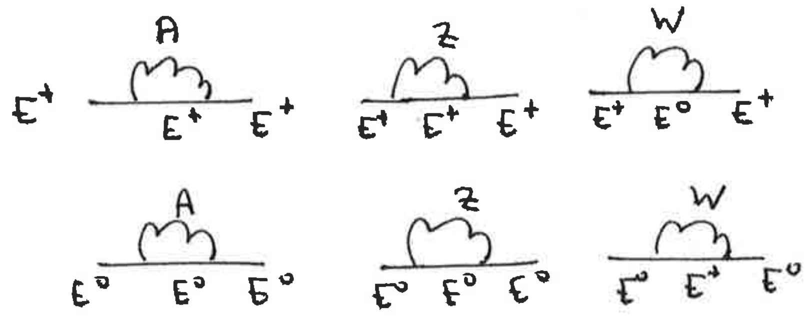


$= i\frac{g}{c_W}\left(\frac{1}{2} - s_W^2\right)$

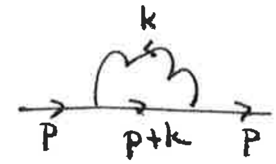


$= i\frac{g}{c_W}\left(-\frac{1}{2}\right)$

The 1-loop diagrams that give the mass correction are:



All of the diagrams have the structure



$$(-ig)^2\cdot(\text{Const})\cdot\int\frac{d^dk}{(2\pi)^d}\,\gamma^\mu\,\frac{i(\not p+\not k+m)}{(p+k)^2-m^2}\,\gamma^\nu\,\frac{-i}{k^2-M^2}\left(g^{\mu\nu}-Ak^\mu k^\nu\right)$$

the $Ak^\mu k^\nu$ term is

$$\not k\,\frac{(\not p+\not k+m)}{(p+k)^2-m^2}\,\not k\,\frac{1}{k^2-M^2}$$

$$=\left[(\not p+\not k-m)-(\not p-m)\right]\frac{(\not p+\not k+m)}{(p+k)^2-m^2}\,\not k\,\frac{1}{k^2-M^2}$$

$$=\left[\frac{(p+k-m)(p+k+m)}{(p+k)^2-m^2}\right]\not k\,\frac{1}{k^2-M^2}-(\not p-m)\,\frac{(\not p+\not k+m)\,\not k}{(p+k)^2-m^2\;k^2-M^2}$$

$$=\underbrace{\not k\,\frac{1}{k^2-M^2}}_{\text{symmetrically integrates to }0}-(\not p-m)\left[\;\cdots\;\right]$$

vanishes on mass shell $\not p=m$

c.) Then we only need to deal with the $g^{\mu\nu}$ part of the vector boson propagator. Now a typical diagram has the structure

$$= (-ig)^2 \, \underline{C} \int \frac{d^d k}{(2\pi)^4} \; \frac{i\,\gamma^\mu(\not p + \not k + M)\,\gamma_\mu}{(p+k)^2 - M^2} \; \frac{-i}{k^2 - m^2}$$

$$\gamma^\mu(\not p + \not k + M)\gamma_\mu = (4-2\epsilon)M - (2-2\epsilon)(\not p + \not k)$$

combine denominators $\quad 2\epsilon = 4-d \quad d = 4-2\epsilon$

$$\frac{1}{(p+k)^2 - M^2}\,\frac{1}{k^2 - m^2} = \int_0^1 dx \, \frac{1}{\left[k^2 + 2x p\cdot k + x p^2 - (xM^2 + (1-x)m^2)\right]^2}$$

$$\mathbb{k} = k + xp \qquad k = \mathbb{k} - xp \qquad (p+k) = \mathbb{k} + (1-x)p$$

$$= \int_0^1 dx \, \frac{1}{[\mathbb{k}^2 - \Delta]^2} \qquad \Delta = xM^2 + (1-x)m^2 - x(1-x)p^2$$

$$= -g^2 C \int_0^1 dx \, \frac{i}{(4\pi)^{d/2}} \, \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[(4-2\epsilon)M - (2-2\epsilon)(1-x)\not p\right]$$

on shell
$\not p = M$

$$= \frac{-ig^2 C}{(4\pi)^{d/2}} \int dx \, \frac{\Gamma(2-d/2)}{[x^2M^2 + (1-x)m^2]^{2-d/2}} \left[4-2\epsilon - (2-2\epsilon)(1-x)\right] M$$

$$= \frac{-ig^2 C}{(4\pi)^2} \int_0^1 dx \left(\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\left[x^2M^2 + (1-x)m^2\right]\right)$$

$$M \cdot \left[(4-2\epsilon) - (2-2\epsilon)(1-x)\right]$$

$$= \frac{-ig^2 C}{(4\pi)^2} \, F(M^2, m^2)$$

the divergent part of F is $\quad F(M^2, m^2) = \frac{1}{\epsilon} M \cdot 3 \quad$ independent of $\underline{m^2}$

then

E^+ 

$$= \frac{-ig^2}{(4\pi)^2}\Big\{ \overset{A}{s_W^2 F(M^2,0)} + \overset{Z}{\frac{1}{c_W^2}\left(\frac{1}{2}-s_W^2\right)^2 F(M^2,m_Z^2)} + \frac{1}{2}F(M^2,m_W^2) \quad W \Big\}$$

E^0 

$$= \frac{-ig^2}{(4\pi)^2}\Big\{ \overset{A}{0} + \overset{Z}{\frac{1}{c_W^2}\left(-\frac{1}{2}\right)^2 F(M^2,m_Z^2)} + \frac{1}{2}F(M^2,m_W^2) \quad W \Big\}$$

the divergent parts are

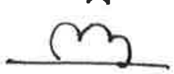
$$E^+: \quad \{\ \} \sim \frac{3M}{\epsilon}\Big\{\frac{1}{c_W^2}\left(\cancel{s_W^2 c_W^2} + \frac{1}{4} - \cancel{s_W^2} + \cancel{s_W^4} + \frac{1}{2}c_W^2\right)\Big\}$$

$$\sim \frac{3M}{\epsilon}\Big\{\frac{1}{c_W^2}\Big\}\left(\frac{3}{4}c_W^2 + \frac{1}{4}s_W^2\right)$$

$$E^0: \quad \{\ \} \sim \frac{3M}{\epsilon}\Big\{\frac{1}{c_W^2}\Big\}\left(\frac{1}{4} + \frac{1}{2}c_W^2\right)$$

$$\sim \frac{3M}{\epsilon}\Big\{\frac{1}{c_W^2}\Big\}\cdot\left(\frac{3}{4}c_W^2 + \frac{1}{4}s_W^2\right)$$

So the divergent parts of the mass shifts are equal. This shouldn't be a surprise. The divergent mass shift in the case of unbroken $SU(2)\times U(1)$ is



$$= \frac{-i}{(4\pi)^2}\left\{ g^2 \cdot \underbrace{\frac{3}{4}}_{(\tau^a\tau^a)} + (g')^2 \left(\frac{1}{2}\right)^2 \right\}$$

$$= \frac{-i}{(4\pi)^2}\, g^2 \left(\frac{3}{4} + \frac{1}{4}\frac{s_w^2}{c_w^2} \right)$$

all divergences should match between the broken and unbroken theories.

Since  $\Big|_{p=M} = -i\,\Delta m$

The relative mass shift is

$$m(E^+) - m(E^0) = \frac{g^2}{(4\pi)^2}\left\{ s_w^2\, F(M^2,0) + \frac{(\frac{1}{2}-s_w^2)^2}{c_w^2} F(M^2,m_Z^2) - \frac{(-\frac{1}{2})^2}{c_w^2} F(M^2,m_Z^2) \right\}$$

$$= \frac{g^2}{(4\pi)^2}\left\{\frac{1}{c_w^2}\right\}\left\{ (s_w^2 - s_w^4)\, F(M^2,0) + (-s_w^2 + s_w^4)\, F(M^2,m_Z^2) \right\}$$

$$= \frac{g^2}{(4\pi)^2}\, s_w^2 \left\{ F(M^2,0) - F(M^2,m_Z^2) \right\}$$

$$m(E^+) - m(E^0) = \frac{g^2}{(4\pi)^2} s_w^2 \, M$$

$$\int_0^1 dx \; 2(1+x) \log\left(\frac{x^2M^2 + (1-x)\, m_z^2}{x^2 M^2}\right)$$

$$= \frac{2e^2}{(4\pi)^2} M \int_0^1 dx \; (1+x) \log\left(1 + \frac{(1-x)}{x^2}\frac{m_z^2}{M^2}\right)$$

d.) Now we need to approximate this integral for $M^2 \gg m_z^2$

Notice that, away from $x=0$, we can expand the log and estimate $\int_A^1 dx \cdots \sim \frac{m_z^2}{M^2}$

This approximation breaks down for $x \sim \frac{m_z}{M}$. But then the integral is approximately

$$\int_0^1 dx \; \log\left(1 + \frac{1}{x^2}\frac{m_z^2}{M^2}\right)$$

$$= \int_0^1 dx \left\{ \log\left(x^2 + m_z^2/M^2\right) - 2\log x \right\}$$

$$= \int_0^1 dx \left[\log\left(x + i\, m_z/M\right) + \log\left(x - i\frac{m_z}{M}\right) - 2\log x \right]$$

$$= \Big[x + i\frac{m_z}{M} \log\left(x + i\frac{m_z}{M}\right) - \left(x + i\frac{m_z}{M}\right)$$
$$+ x - i\frac{m_z}{M}\log\left(x - i\frac{m_z}{M}\right) - \left(x - i\frac{m_z}{M}\right)$$
$$- 2x\log x + 2x \Big]\Big|_0^1$$

$$= \left(1+i\frac{m_Z}{M}\right)\log\left(1+i\frac{m_Z}{M}\right) + \left(1-i\frac{m_Z}{M}\right)\log\left(1-i\frac{m_Z}{M}\right) - 2\cdot 0$$

$$- \left(1+i\frac{m_Z}{M}\right) - \left(1-i\frac{m_Z}{M}\right) + 2$$

$$- i\frac{m_Z}{M}\log\left(i\frac{m_Z}{M}\right) + i\frac{m_Z}{M}\log\left(-i\frac{m_Z}{M}\right) + 2\cdot 0$$

$$= -i\frac{m_Z}{M}\; i\frac{\pi}{2} + i\frac{m_Z}{M}\cdot\left(-i\pi/2\right) = \frac{\pi m_Z}{M}$$

then

$$m(E^+) - m(E^0) = \frac{e^2}{4\pi}\cdot\frac{1}{2}\, m_Z$$

$$= \frac{1}{2}\alpha\, m_Z = \underline{356\ \text{MeV}}$$

91.187 GeV

actually one should put $\alpha = \frac{1}{128}$ due to running

so $\Delta m_E > m_{\pi^+}$ but not by much!

The decay $E^+ \to E^0 + K^+$ is forbidden energetically.

$E^+ \to E^0 + 2\pi$ is just barely possible, but the phase space is very small.

e.) Now repeat this for $\Sigma^+ \Sigma^0$. The kinetic part of the Lagrangian is

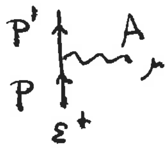
$$\mathcal{L} = \left| \partial_\mu \begin{pmatrix}\Sigma^+\\ \Sigma^0\end{pmatrix} - i\frac{g}{\sqrt{2}} W^+_\mu I^+ \begin{pmatrix}\Sigma^+\\ \Sigma^0\end{pmatrix} - i\frac{g}{\sqrt{2}} W^-_\mu I^- \begin{pmatrix}\Sigma^+\\ \Sigma^0\end{pmatrix} \right.$$

$$\left. - ie A_\mu \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Sigma^+\\ \Sigma^0\end{pmatrix} - i\frac{g Z_\mu}{c_W}\begin{pmatrix}\frac{1}{2}-s_W^2 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\Sigma^+\\ \Sigma^0\end{pmatrix} \right|^2$$

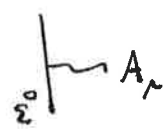
$$= \left| \partial_\mu \Sigma^+ - \frac{ig}{\sqrt{2}} W^+_\mu \Sigma^0 - ieA_\mu \Sigma^+ - i\frac{g}{c_W} Z_\mu \left(\tfrac{1}{2}-s_W^2\right)\Sigma^+ \right|^2$$

$$+ \left| \partial_\mu \Sigma^0 - i\frac{g}{\sqrt{2}} W^-_\mu \Sigma^+ - i\frac{g}{c_W} Z_\mu \left(-\tfrac{1}{2}\right)\Sigma^0 \right|^2$$

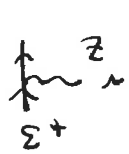
The Feynman rules are



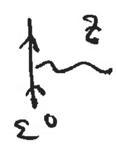
$ie(p+p')^\mu$



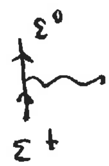
$= 0$



$i\frac{g}{c_W}\left(\frac{1}{2}-s_W^2\right)(p+p')^\mu$



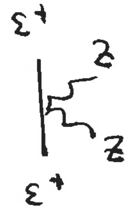
$-i\frac{g}{c_W}\frac{1}{2}(p+p')^\mu$



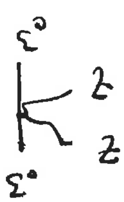
$i\frac{g}{\sqrt{2}}(p+p')^\mu$

$i\frac{g}{\sqrt{2}}(p+p')^\mu$

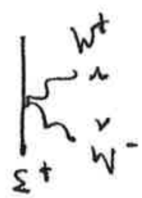
$2ie^2 g^{\mu\nu}$



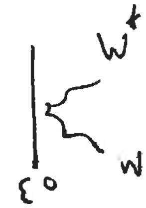
$2i\frac{g^2}{c_W^2}\left(\frac{1}{2}-s_W^2\right)^2 g^{\mu\nu}$



$2i\frac{g^2}{c_W^2}\frac{1}{4}g^{\mu\nu}$

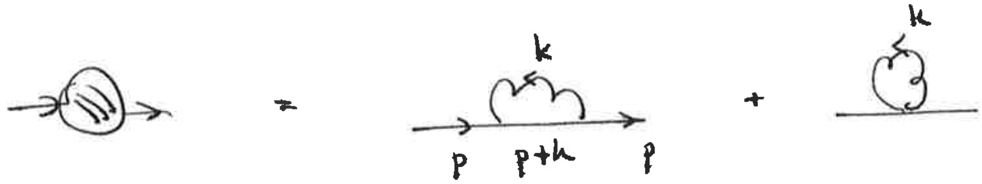


$\frac{ig^2}{2} g^{\mu\nu}$



$\frac{ig^2}{2} g^{\mu\nu}$

A representative self-energy contribution is



$$= \int \frac{d^d k}{(2\pi)^d} \Big\{ (ig)^2 \underline{C} \quad (2p+k)^\mu \frac{i}{(p+k)^2 - M^2} (2p+k)^\nu$$

$$+ \frac{1}{2} \cdot 2 i g^2 \underline{C} \quad g^{\mu\nu} \Big\}$$

$$\frac{-i}{k^2 - M^2} \left(g_{\mu\nu} - A k_\mu k_\nu \right)$$

Work on the $k_\mu k_\nu$ term.

$$\{\} = -ig^2 \underline{C} \left[\frac{k\cdot(2p+k) \quad k\cdot(2p+k)}{(p+k)^2 - M^2} - k^2 \right]$$

$$k\cdot(2p+k) = k^2 + 2p\cdot k = \left((p+k)^2 - M^2\right) - (\underbrace{p^2 - M^2}_{0 \text{ on shell}})$$

$$= -ig^2 \underline{C} \quad \left[(k\cdot 2p + k^2) - k^2 \right]$$

$$= -ig^2 C \quad k\cdot 2p$$

vanishes after symmetric integration over $\underline{k}$

Now, the $g^{\mu\nu}$ term gives

$$(-ig^2\underline{C})\,(-i)\int\frac{d^dk}{(2\pi)^d}\left[\frac{(2p+k)^2}{((p+k)^2-M^2)(k^2-m^2)}-\frac{d}{k^2-m^2}\right]$$

$$= -g^2\underline{C}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^4}\left\{\frac{(2p+k)^2}{[\ell^2-\Delta]^2}-\frac{d}{\ell^2-m^2}\right\}$$

$$\Delta = xM^2+(1-x)m^2-x(1-x)p^2$$

$$k = \ell - xp$$

$$k+2p = \ell+(2-x)p$$

$$= -g^2\underline{C}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^4}\left\{\frac{\ell^2+(2-x)^2p^2}{[\ell^2-\Delta]^2}-\frac{d}{\ell^2-m^2}\right\}$$

$$= -g^2\underline{C}\int_0^1 dx\,\frac{i}{(4\pi)^{d/2}}\left\{-\frac{d/2\,\Gamma(1-d/2)}{[\Delta]^{1-d/2}}+\frac{d\,\Gamma(1-d/2)}{[m^2]^{1-d/2}}\right.$$

$$\left.+\frac{(2-x)^2p^2\,\Gamma(2-d/2)}{[\Delta]^{2-d/2}}\right\}$$

There is a pole at $d=2$, but it is independent of all masses. The coefficient is

$$\varepsilon^+ \quad g^2\left[s_w^2+\frac{1}{c_w^2}\left(\tfrac{1}{2}-s_w^2\right)^2+\frac{1}{2}\right]$$

A

$$\varepsilon^0 \quad g^2\left[\frac{1}{c_w^2}\left(-\tfrac{1}{2}\right)^2+\frac{1}{2}\right]$$

We have seen this before. Both equal $g^2\frac{3}{4}+(g')^2\frac{1}{4}$

and so the pole at $d = 2$ cancels between

 Σ^+ and  Σ^0

Next, look at the pole at $d = 4$

$$= -g^2 C \int_0^1 dx \, \frac{i}{(4\pi)^{d/2}} \left\{ -\frac{d/2}{1-d/2} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \Delta \right.$$

$$+ \frac{d}{1-d/2} \frac{\Gamma(2-d/2)}{[m^2]^{2-d/2}} m^2$$

$$\left. + \Gamma(2-d/2) \frac{(2-x)^2 p^2}{\Delta^{2-d/2}} \right\}$$

The pole is

$$\Gamma(2-d/2) \left\{ 2 \cdot (x M^2 + (1-x) m^2 - x(1-x) p^2) \right.$$

$$\left. - 4 \quad m^2 \quad + \quad (2-x)^2 p^2 \right\}$$

The terms that do not depend on m^2 cancel between Σ^+ and Σ^0 as before. For the terms proportional to m^2, these are proportional to

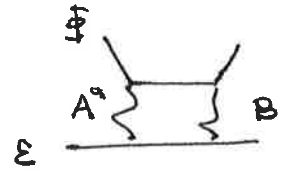
$$\Sigma^+ \qquad g^2 \Big\{ \underset{A}{0} + \underset{Z}{\frac{1}{c_w^2}\left(\tfrac{1}{2} - s_w^2\right)^2 \cdot \frac{g^2}{4 c_w^2}} + \underset{W}{\frac{1}{2} \cdot \frac{g^2}{4}} \Big\}$$

$$\Sigma^0 \qquad g^2 \Big\{ 0 + \frac{1}{c_w^2} \frac{1}{4} \frac{g^2}{4 c_w^2} + \frac{1}{2} \frac{g^2}{4} \Big\}$$

These are $\underline{\underline{\underline{\text{not}}}}$ equal, and so

$$m(\Sigma^+) - m(\Sigma^0) = \underline{\text{log divergent}}$$

This is surprising, but also understandable from the point of view of the symmetric theory. In that theory diagrams such as



$\leftarrow \quad D = 0$

can generate the operator

$$\delta\mathcal{L} = |\Sigma^\dagger \Phi|^2 \times \text{log divergent coefficient}$$

this operator is invariant under all of the symmetries of the original Lagrangian, so it should have been included. Note that, when the symmetry is broken

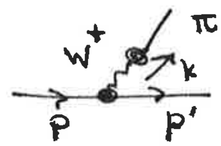
$$\Sigma^\dagger \Phi \to \frac{1}{\sqrt{2}} v \Sigma^0$$

so $\quad \delta\mathcal{L} = -c\,|\Sigma^\dagger \Phi|^2 \quad$ gives a contribution

$$\delta\mathcal{L} = -\frac{c}{2} v^2 |\Sigma^0|^2$$

to the $\Sigma^+ - \Sigma^-$ mass difference. The coefficient $\underline{c}$ is not computable within the theory.

f.) The decay of the $E^+ \to E^0 \pi^+$ is given by



$$i\mathcal{M} = -\frac{ig}{\sqrt{2}}\bar{u}(E^0)\,\gamma^\mu\, u(E^+)\;\frac{-i}{k^2 - m_W^2}\left(-i\,\frac{k^\mu}{2}\, f_\pi \sqrt{2}\right)$$

(since $\langle 0|j^{\mu 1}|\pi^1\rangle = \langle 0|j^2|\pi^2\rangle = i\,\frac{P^\mu}{\cancel{2}}\, f_\pi$

$$\langle 0|\, j^{\mu 1} - i j^{\mu 2} \,\Big|\, \frac{\pi^1 + i\pi^2}{\sqrt{2}}\Big\rangle = i\,\frac{P^\mu}{2}\, f_\pi \cdot \sqrt{2}\ )$$

$$k^\mu\ \bar{u}(E^0)\,\gamma_\mu\, u(E^+)$$

$$= \bar{u}\ \not{k}\ u$$

$$= \bar{u}\ (\not{p} - \not{p}')\ u$$

$$= (m_{E^+} - m_{E^0})\ \bar{u}u$$

$$= (m_{E^+} - m_{E^0})\ 2M\ \xi'^\dagger \xi$$

For $M \gg m_\pi$, the kinematics is easy! the E^0 is near rest and the energy of the π is $m_{E^+} - m_{E^0}$

$$\mathcal{M} = \left(i\,\frac{g}{\sqrt{2}}\right)^2 2M\ (m_{E^+} - m_{E^0})\ \xi^\dagger \xi\ \underset{\substack{\uparrow \\ k \approx 0}}{\frac{1}{-m_W^2}}\ \frac{1}{\sqrt{2}}\, f_\pi$$

$$\sum_{s'} |M|^2 = \frac{g^4 M^2}{2 m_W^4} f_\pi^2 (m_{E^+} - m_{E^0})^2$$

$$\Gamma = \frac{1}{2M} \int d\Pi_2 \sum_{s'} |M|^2$$

$$\int d\Pi_2 = \frac{1}{8\pi} \frac{2k}{M}$$

k = momentum of the π

$$= (m_E^\bullet - m_{E^0}^\bullet)^2 - m_\pi^2$$

$$\frac{g^4}{m_W^4} = \frac{g^4}{g^4 v^4 / 2^4} = \frac{16}{v^4} \qquad v = 246 \text{ GeV}$$

$$\Gamma = \frac{1}{2M} \frac{1}{4\pi} \frac{k}{M} \frac{8}{v^4} M^2 f_\pi^2 (m_{E^+} - m_{E^0})^2$$

$$= \frac{1}{\pi} \frac{f_\pi^2 k (m_{E^+} - m_{E^0})^2}{v^4}$$

Now. $v = 246$ GeV

$f_\pi = 93$ MeV for the real pion

$(m_{E^+} - m_{E^0}) = 356$ MeV

$k = 327$ MeV

so $\Gamma = 3.1 \times 10^{-11}$ MeV $\div$ $\overset{\hbar c}{197 \text{ MeV fm}}$

$1/\Gamma = 6.3 \times 10^{12}\ 10^{-13}$ cm

$c\tau = \underline{0.63 \text{ cm}}$ a macroscopic distance