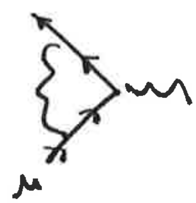
Physics 331 - Final Exam

Solutions

1.) a.) $a_\mu = F_2(q^2 = 0)$

In 1-loop, this is computed from

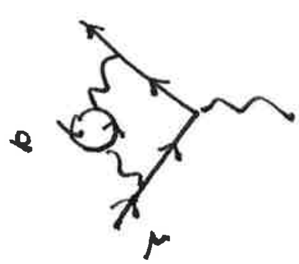


The calculation is exactly the same as for the electron (g-2). The answer is dimensionless

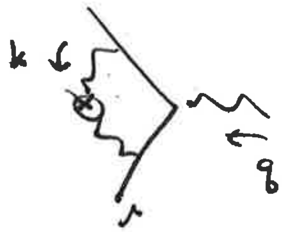
$$a = \frac{\alpha}{2\pi}$$

So $a_e = a_\mu$. to this order

b.) The first electron contribution to the muon g-2 comes from



The subgraph $\underset{e}{\sim\bigcirc\sim}$ is the electron vacuum polarization diagram. This diagram is divergent. However, renormalized perturbation theory also includes the diagram



where  is the photon self-energy counterterm. We computed in class

k  +  =

$$i\,(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(-\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\;\log\left[\frac{m_e^2}{m_e^2 - x(1-x)\,k^2}\right]\right)$$

then

 $\sim \frac{1}{k^2}\, k^2 \log k^2\, \frac{1}{k^2} \sim k^2 \log k^2$

c.) This gives a divergent contribution to F_1 but a finite contribution to F_2. In fact, we can estimate

$$\log\left(\frac{m_e^2}{m_e^2 - x(1-x)\,k^2}\right) \approx \log\frac{m_e^2}{m_\mu^2}$$

since momenta of order m_μ dominate the loop. Then

 $= \frac{-i}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\frac{\alpha}{3\pi}\log m_\mu^2/m_e^2$

the $k^\mu k^\nu$ term gives 0 by the Ward identity. Then the computation of a_μ is identical to the 1-loop one and we

find
$$a_\mu = \frac{\alpha}{2\pi}\left(1 + \frac{2\alpha}{3\pi}\log m_\mu/m_e + \cdots\right)$$
where $\ldots$ are terms of order α without the logarithmic enhancement.

d.) For the muon contribution to a_e, there is a similar story. The leading effect comes from

leads to a propagator correction for the photon

$$\left(\frac{-i}{k^2} g^{\mu\nu}\right)\left(\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\, \log\frac{m_\mu^2 - x(1-x)k^2}{m_\mu^2}\right)$$

However, the loop is now dominated by momenta of order $m_e \ll m_\mu$. So this becomes

$$\frac{-i}{k^2} g^{\mu\nu}\, \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\, \left(x(1-x)\frac{(-k^2)}{m_\mu^2} + \cdots\right)$$

with $(-k^2) \sim m_e^2$. Then the correction to $F_2(0)$ is of order

$$\left(\frac{m_e^2}{m_\mu^2}\right) \times (\text{possible logs of } m_\mu/m_e)$$

The coefficient of this term is nontrivial to obtain; you really have to analyze the 2-loop diagram in detail. The result is

$$a_e = \frac{\alpha}{2\pi} + \frac{1}{45}\left(\frac{\alpha}{\pi}\right)^2 \frac{m_e^2}{m_\mu^2} + \cdots$$

e.) Why are these answers so different. Both effects come from the running of α, which gives a propagator $\propto (\text{coupling})^2$

$$-i \frac{\alpha(k^2)}{k^2} g^{\mu\nu}$$

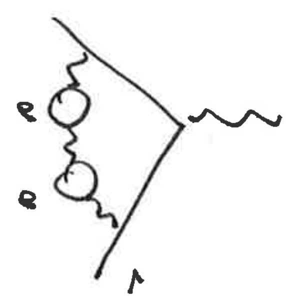
$\alpha(0) = \frac{1}{137}$. At $k^2 = \mu^2$, there is substantial running due to the electron.

$$\alpha(m_\mu^2) = \frac{\alpha(0)}{1 - \frac{2\alpha(0)}{3\pi} \log m_\mu / m_e}$$

On the other hand, the muon contribution to the running has not yet taken off at $k^2 \sim m_e^2$

$$\alpha(m_e^2) = \frac{\alpha(0)}{1 - \Pi_e(k^2) - \underbrace{\Pi_\mu(k^2)}_{\text{this is } \sim \frac{m_e^2}{m_\mu^2}} \Big|_{k^2 = m_e^2}}$$

f.) The diagrams that give one $\log m_\mu/m_e$ for each factor of α are



and the associated counterterm diagrams. These give:

$$= \frac{-i}{k^2}\left(1 + \frac{2\alpha}{3\pi}\log m_\mu/m_e + \left(\frac{2\alpha}{3\pi}\right)^2\left(\log m_\mu/m_e\right)^2 + \cdots\right)$$

$$= \frac{-i}{k^2}\,\frac{1}{1 - \frac{2\alpha}{3\pi}\log m_\mu/m_e}$$

so to this order

$$a_\mu = \frac{\alpha}{2\pi}\,\frac{1}{1 - \frac{2\alpha}{3\pi}\log m_\mu/m_e}$$

g) If we would only like to count factors of log m_μ/m_e, we could adopt an off-shell renormalization scheme in which we renormalize at the scale M. Then

$$\left[M\frac{\partial}{\partial M} + \beta(e)\frac{\partial}{\partial e}\right] a_\mu = 0$$

There is no γ term, because the normalization of a_μ is fixed since a_μ is directly observable. Now, $\alpha(M) = \alpha(0) +$ terms of order $\alpha^2 \cdot 1$. So setting $M = m_e$ accounts all logarithms

With $\beta(e) = \frac{e^3}{12\pi^2}$

$$\left[M\frac{\partial}{\partial M} + \beta(e)\frac{\partial}{\partial e}\right] a_\mu = 0$$

is solved by

$$a_\mu = a_\mu(e(m_\mu))$$

where

$$\frac{de}{d\log M'} = \beta(e(M')) \qquad e(M) = e$$

The solution of the RG equation is

$$e^2(M') = \frac{e^2}{1 - \frac{e^2}{6\pi} \log M'/M}$$

So we find, setting $M = m_e$ $\quad M' = m_\mu$

$$a_\mu = a_\mu\left(\frac{\alpha}{1 - \frac{2}{3\pi}\alpha \log m_\mu/m_e} \right)$$

From the data $\quad a_\mu = \frac{\alpha}{2\pi} + O(\alpha^2)$ $\quad$ we have

$$a_\mu = \frac{1}{2\pi} \frac{\alpha}{1 - \frac{2}{3\pi}\alpha \log m_\mu/m_e} \quad \checkmark$$

h) Now write a_μ in the form

$$a_\mu = A_1 \alpha + A_{2,1} \alpha^2 \log m_\mu/m_e + A_{2,0} \alpha^2$$
$$+ A_{3,2} \alpha^3 (\log m_\mu/m_e)^2 + A_{3,1} \alpha^3 \log m_\mu/m_e + A_{30}\alpha^3$$
$$+ \cdots$$

the C-S equation tells us that $\left[\frac{\partial}{\partial \lg m_e} + \beta(e)\frac{\partial}{\partial e}\right](a_\mu) = 0$

$$-\frac{\partial}{\partial \lg m_e} a_\mu = A_{2,1}\,\alpha^2 + 2A_{3,2}\,\alpha^3 \lg m_\mu/m_e$$

$$+ A_{3,1}\,\alpha^3$$

$$\beta(e)\frac{\partial}{\partial e} = 2e\,\beta(e)\frac{\partial}{\partial e^2} = 2(4\pi)^{-1} e\,\beta(e)\frac{\partial}{\partial \alpha}$$

$$= \left(\frac{8}{3}\frac{e^4}{(4\pi)^3} + 8\frac{e^6}{(4\pi)^5} + \cdots\right)\frac{\partial}{\partial \alpha}$$

$$= \left(\frac{8}{3}\frac{\alpha^2}{4\pi} + 8\frac{\alpha^3}{(4\pi)^2} + \cdots\right)\frac{\partial}{\partial \alpha}$$

$$\beta(e)\frac{\partial}{\partial e} = A_1\left(\frac{2}{3\pi}\alpha^2 + \frac{1}{2\pi^2}\alpha^3\right)$$

$$+ 2\left(A_{2,1}\lg m_\mu/m_e + A_{2,0}\right)\left(\frac{8}{3}\frac{\alpha^3}{4\pi} + \cdots\right)$$

$$+ \cdots$$

Match terms:

$$A_{2,1} = A_1\frac{2}{3\pi}$$

$$2A_{3,2} = 2A_{2,1}\frac{2}{3\pi}$$

$$A_{3,1} = A_1\frac{1}{2\pi^2} + A_{2,0}\frac{4}{3\pi}$$

This gives $A_{2,1} = \frac{2}{3\pi}\cdot\frac{1}{2\pi}$ $A_{3,2} = \left(\frac{2}{3\pi}\right)^2\frac{1}{2\pi}$, which we know already, plus

$$A_{3,1} = \frac{1}{2\pi}\frac{1}{2\pi^2} + \frac{4}{3\pi} A_{2,0}$$

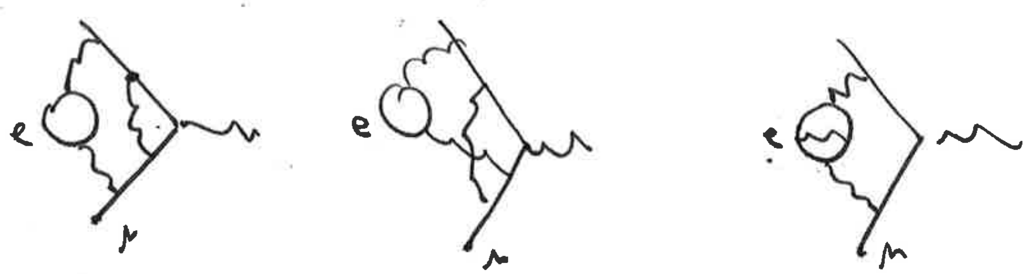
$$= \frac{1}{4\pi^3} + \frac{4}{3\pi}\frac{1}{\pi^2} A_2$$

then

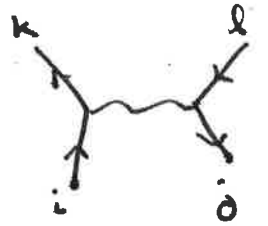
$$a_\mu = \frac{1}{2\pi}\left(\frac{\alpha}{1-\frac{2}{3\pi}\alpha \lg m_\mu/m_e}\right)\left(1 + \left(\frac{1}{2\pi^2} + \frac{4}{3\pi^2}A_2\right)\alpha^2 \lg\frac{m_\mu}{m_e}\right)$$

$$= \quad \cdots \quad + \underbrace{\left(\frac{1}{4} + \frac{4}{3}A_2\right)}_{\simeq 0.188}\left(\frac{\alpha}{\pi}\right)^3 \lg m_\mu/m_e$$

some diagrams that contribute to this term are:



2.) a) In leading order in QCD, quark-antiquark scattering in the nonrelativistic region is given by



$$= \underbrace{ig(\bar{u}\gamma^0 u)}_{2m_q}\left(\frac{-i g^{00}}{q^2}\right)\underbrace{\bar{v}\gamma^0 v}_{-2m_q}\, ig$$

$$\cdot\ t^a_{ki}\, t^a_{jl}$$

$$= -i(2m_q)^2\,\frac{g^2}{q^2}\, t^a_{ki}\, t^a_{jl} = -i(2m_q)^2\,\frac{-g^2}{|\vec{q}|^2}\, t^a_{ki}\, t^a_{jl}$$

on a color wavefunction $\propto \delta_{ij}$

$$t^a_{ki}\, t^a_{jl} \rightarrow (t^a t^a)_{kl} = C_2(R)\,\delta_{kl}$$

so

$$V = -C_2(\underline{3})\,\frac{g^2}{|\vec{q}|^2} \qquad C_2(\underline{3}) = \frac{4}{3} \quad *$$

In higher orders, V obeys the Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right] V(q, g, M) = 0$$

V is a physically measurable energy, so its scale is fixed and there

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* $\quad \mathrm{tr}\, t^a t^a = \frac{1}{2}\delta^{aa} = \frac{8}{2} = \mathrm{tr}\, C_2(\underline{3})\cdot 1 = 3\, C_2(\underline{3})$

$$\Rightarrow \quad C_2(\underline{3}) = \frac{4}{3}$$

is no γ term. Then

$$V = V(q, g(q)) \quad \text{with} \quad \frac{d}{d\log M'} g(M') = \beta(g(M'))$$

b.) The leading-order QCD beta function is

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(R)\right]$$

$$= -\frac{g^3}{(4\pi)^2}\left[11 - \frac{2}{3}n_f\right]$$

Let $\beta(g) = -\frac{g^3}{(4\pi)^2}\cdot b$

where $b = \frac{23}{3}$ (5 flavors) ; $\frac{21}{3}$ (6 flavors)

Then $\frac{dg}{d\log q} = \beta(g(q)) = -\frac{g^3}{(4\pi)^2} b$

$$\frac{d\alpha_s}{d\log q} = -\frac{b}{2\pi}\alpha_s^2$$

$$\alpha_s(q) = \frac{\alpha_s(M)}{\left(1 + \frac{b}{2\pi}\alpha_s(M)\log q/M\right)}$$

This expression goes to ∞ at

$$-1 = \frac{b}{2\pi}\alpha_s(M)\log q/M$$

$$q = M\exp\left[-\frac{2\pi}{b\alpha_s(M)}\right]$$

This defines Λ.

setting $M = 91$ GeV $\quad \alpha_s(M) = 0.118 \quad b = \frac{23}{3}$

$$\Lambda = 88 \text{ MeV}$$

c.) The t quark contributes to the QCD potential through the diagrams

$$= (\text{above result}) \cdot \left[i(q^2 g^{\mu\nu})\, \Pi(q^2) \right] \left(\frac{-i}{q^2} \right)$$

where

$$\Pi(q^2) = \underbrace{\frac{1}{2}}_{C(R)} \cdot \left(-\frac{2\alpha_s}{\pi} \right) \int_0^1 dx\; x(1-x) \left\{ \log\left(\frac{*}{m_t^2 - x(1-x) q^2} \right) - (-\otimes-) \right\}$$

where the subtraction makes this term zero at $q^2 = M^2 \gg m_t^2$

Then

$$\Pi(q^2) = -\frac{\alpha_s(M)}{\pi} \int_0^1 dx\; x(1-x) \log\left[\frac{+x(1-x) M^2}{m_t^2 + x(1-x)\, |\vec{q}|^2} \right]$$

Including also the effect of α_s running with gluons + 5 flavors, as described on p. 10, we have

$$V(q) = -\frac{4}{3} \frac{1}{|\vec{q}|^2}\, g^2(q)$$

Set $\alpha_s(q) = g^2(q)/4\pi$; then

$$\alpha_s(q) = \frac{\alpha_s(M)}{\left(1 + \frac{b\,\alpha_s(M)}{2\pi}\log q/M + \frac{\alpha_s(M)}{\pi}\int_0^1 dx\, x(1-x)\,\log\left[\frac{m_t^2 + x(1-x)M^2}{m_t^2 + x(1-x)q^2}\right]\right)}$$

$b = \frac{23}{3}$ (5-flavor running)

d) For $|\vec{q}|^2 \gg m_t^2$ this function is

$$\alpha_s(q) = \frac{\alpha_s(M)}{1 + \frac{b\,\alpha_s(M)}{2\pi}\log q/M + \frac{\alpha_s(M)}{\pi}\frac{1}{6}\cdot\log M^2/q^2}$$

$$= \frac{\alpha_s(M)}{1 + \frac{\alpha_s(M)}{2\pi}\underbrace{\left(\frac{23}{3} - \frac{2}{3}\right)}_{21/3}\log q/M}$$

this is running with 6 flavors. On the other hand, for $|\vec{q}|^2 \ll m_t^2$

$$\alpha_s(q) = \frac{\alpha_s(M)}{1 + \frac{b}{2\pi}\alpha_s(M)\log\frac{q}{M} + \frac{\alpha_s(M)}{\pi}\int_0^1 dx\, x(1-x)\,\log\frac{x(1-x)M^2}{m_t^2}}$$

Evaluate the integral:

$$\int_0^1 dx\, x(1-x)\,\log\frac{x(1-x)M^2}{m_t^2} = \frac{1}{6}\log\left(\frac{M^2}{m_t^2}\right) + \int_0^1 dx\, x(1-x)\,\log x(1-x)$$

$$= \frac{1}{6}\log M^2/m_t^2 + 2\int_0^1 dx\,(x-x^2)\log x$$

$$= \frac{1}{6}\log M^2/m_t^2 + 2\left\{ \frac{x^2}{2}\log x - x^2/4 - \frac{x^3}{3}\log x + \frac{x^3}{9}\right\}\Big|_0^1$$

$$= \frac{1}{6}\log M^2/m_t^2 + 2\left[-\frac{1}{4}+\frac{1}{9}\right]$$

$$= \frac{1}{6}\left(\log M^2/m_t^2 - 5/3\right)$$

Taking the inverse of the formula above, we can express 5-flavor running as

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} + \frac{b}{2\pi}\log\frac{q}{M}$$

The formula we have now is

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} + \frac{b}{2\pi}\log q/M + \frac{1}{6\pi}\left(\log\frac{M^2}{m_t^2} - 5/3\right)$$

$$= \frac{1}{\tilde{\alpha}(M)} + \frac{b}{2\pi}\log q/M$$

so this is 5-flavor running with a different initial condition

$$\tilde{\alpha}(M) = \left[\frac{1}{\alpha_s(M)} + \frac{1}{6\pi}\left(\log M^2/m_t^2 - 5/3\right)\right]^{-1}$$

Model this by running with 5 or 6 <u>massless</u> quarks with a discontinuity at $q = C m_t$. For $q > C m_t$

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} + (b - 2/3)\frac{1}{2\pi}\log\frac{q}{M}$$

At $q = C m_t$

$$\frac{1}{\alpha_s(Cm_t)} = \frac{1}{\alpha_s(M)} + (b - 2/3)\frac{1}{2\pi}\log\frac{Cm_t}{M}$$

Then below $q = Cm_t$

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} + (b-2/3)\frac{1}{2\pi}\log\frac{Cm_t}{M} + \frac{b}{2\pi}\log\frac{q}{Cm_t}$$

$$= \frac{1}{\alpha_s(M)} - \frac{1}{3\pi}\log\frac{Cm_t}{M} + \frac{b}{2\pi}\log\frac{q}{M}$$

This reproduces the formula on p. 13 if we set

$$-\frac{1}{3\pi}\log\frac{Cm_t}{M} = \frac{1}{6\pi}\left(\log M^2/m_t^2 - 5/3\right)$$

$$\log\frac{m_t}{M} + \log C = \log\frac{m_t}{M} + \frac{5}{6}$$

so the correct matching point is at $C = \exp[5/6]$

or at $m_t \cdot e^{5/6} = 2.3\, m_t$

e) Using

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} - \frac{1}{3\pi}\left(\log\frac{m_t}{M} + \frac{5}{6}\right) + \frac{b}{2\pi}\log\frac{q}{M}$$

$\Lambda = q$ is the point where this expression vanishes.

Then $\Lambda = M \exp\left[-\frac{2\pi}{b\alpha_s(M)} + \frac{2}{3b}\left(\log\frac{m_t}{M} + \frac{5}{6}\right)\right]$

then

$$\left.\frac{\partial \Lambda}{\partial m_t}\right|_{M,\, \alpha_s(M)} = \Lambda \cdot \frac{2}{3b}\,\frac{1}{m_t}$$

$$= \frac{2}{23}\left(\frac{\Lambda}{m_t}\right)$$

f.) Similarly $\quad \dfrac{\partial m_p}{\partial m_t} = \dfrac{2}{23}\dfrac{m_p}{m_t}$

g.) $\quad g_{pph} = \dfrac{2}{23}\dfrac{m_p}{\langle h\rangle} = 3.3\times 10^{-4}$

h.) Repeat this argument to decouple the b quark. The expression at the start of part (e) would be

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} - \frac{1}{3\pi}\left(\log\frac{m_t}{M} + A\right) - \frac{1}{3\pi}\left(\log\frac{m_t}{M} + B\right)$$
$$+ \frac{1}{2\pi}\, b_4 \log\frac{q}{M}$$

where $b_4 = \dfrac{25}{3} = \left(11 - \dfrac{2}{3}\cdot 4\right)$

Similarly, if we regard $2m_c \gg m_p$ we have

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(M)} - \frac{1}{3\pi}\left[\log\frac{m_t}{M} + \log\frac{m_b}{M} + \log\frac{m_c}{M} + (\text{const})\right]$$
$$+ \frac{1}{2\pi}\, b_3 \log\frac{q}{M}$$

with $b_3 = 11 - \frac{2}{3}\cdot 3 = 9$

on the other hand, we can treat m_u, m_d, m_s directly by perturbation theory using

$$\delta\mathcal{L} = -m_u \bar{u}u - m_d \bar{d}d - m_s \bar{s}s$$

This gives only small corrections to m_p. Then

$$g_{pph} \cong 3 \cdot \frac{2}{27} \frac{m_p}{\langle h \rangle}$$

$$\cong 8.3 \times 10^{-4}$$

c.) $$\alpha_{h} = \frac{g_{pph}^2}{4\pi} = 6 \times 10^{-8}$$

An estimate of the scattering cross section from Higgs exchange is

$$\sigma \cong \frac{\pi \alpha_h^2}{m_p^2} = 1.0 \times 10^{-14}\ \mathrm{GeV}^{-2}$$

$$\times\ 0.389\ \mathrm{GeV}^2\ \mathrm{mb}$$

$$\cong 0.4 \times 10^{-17}\ \mathrm{barn}$$

$$\sigma \cong 0.4 \times 10^{-41}\ \mathrm{cm}^2$$