Physics 331 - Final Exam Solutions

(a) (1) $\psi_i^T c \psi_j = -\psi_{j\alpha}(\psi_i^T c)_\alpha$ (Fermi statistics)

$$= -\psi_j^T c^T \psi_i = +\psi_j^T c \psi_i$$

so only the symmetric part of m_{ij} contributes

(2) $(\psi_i^\dagger i\bar\sigma^\mu \partial_\mu \psi_i)^\dagger = \partial_\mu \psi_i^\dagger \, -i(\bar\sigma^\mu)^\dagger \psi_i$

$$= \partial_\mu \psi_i^\dagger (-i)\bar\sigma^\mu \psi_i = +\psi_i^\dagger i\bar\sigma^\mu \partial_\mu \psi_i$$ (integrate by parts)

$$(m_{ij}\psi_i^T c \psi_j)^\dagger = \psi_j^\dagger (c)^\dagger \psi_i^* \, m_{ij}^\dagger$$

$$= -\psi_j^\dagger c \psi_i^* \, m_{ij}^*$$ since (m_{ij} is symmetric)

(3) Under a Lorentz transformation

$$\psi \to (1 + i\vec\theta\cdot\tfrac{\vec\sigma}{2} - \vec\beta\cdot\tfrac{\vec\sigma}{2})\psi$$

then

$$\psi_i^\dagger \sigma^0 \psi_i = \psi_i^\dagger \psi_i \to \psi_i^\dagger (1 - 2\vec\beta\cdot\tfrac{\vec\sigma}{2})\psi_i$$

$$= \psi_i^\dagger \psi_i - \vec\beta\cdot\psi_i^\dagger\vec\sigma\psi_i$$

$$\psi_i^\dagger \vec\sigma^k \psi_i \to \psi_i^\dagger (1 - i\vec\theta\cdot\tfrac{\vec\sigma}{2} - \beta\tfrac{\vec\sigma}{2})\vec\sigma^k(1 + i\vec\theta\cdot\tfrac{\vec\sigma}{2} - \beta\tfrac{\vec\sigma}{2})\psi_i$$

$$= \psi_i^\dagger \vec\sigma^k \psi_i - i\tfrac{\theta^i}{2}\psi_i^\dagger[\vec\sigma^i, \vec\sigma^k]\psi_i - \beta\psi_i^\dagger\psi_i$$

$$= \psi_i^\dagger \vec\sigma^k \psi_i + \theta^i \epsilon^{ikl}\psi_i^\dagger \vec\sigma^l \psi_i - \beta\psi_i^\dagger \sigma^0 \psi_i$$

then

$$\psi_i^\dagger \bar\sigma^\mu \psi_i = \psi_i^\dagger(\sigma^0, -\sigma^k)\psi_i$$

$$\rightarrow \psi_i^\dagger \bar\sigma^\mu \psi_i + (\vec\beta^k \cdot \psi_i^\dagger \bar\sigma^k \psi_i \, , \, \epsilon^{klm}\theta^l \psi^\dagger \bar\sigma^l \psi + \beta^k \psi^\dagger \bar\sigma^0 \psi)$$

so

$\psi_i^\dagger \bar\sigma^\mu \psi_i$ transform like a 4-vector

$$\psi_i^T c \psi_j \rightarrow \psi_i^T (1 + i\vec\theta \cdot \frac{\vec\sigma}{2}^T - \vec\beta \frac{\vec\sigma}{2}^T) c (1 + i\vec\theta \cdot \frac{\vec\sigma}{2} - \vec\beta \frac{\vec\sigma}{2})\psi_j$$

now $(\vec\sigma^k)^T c = \begin{cases} k=1,3 & +\sigma^k c = -c\sigma^k \\ k=2 & -\sigma^k c = -c\sigma^k \end{cases}$

so

$$\psi_i^T c \psi_j \longrightarrow \psi_i^T c \, (1 - i\vec\theta \frac{\vec\sigma}{2} + \vec\beta \frac{\vec\sigma}{2})(1 + i\vec\theta \cdot \frac{\vec\sigma}{2} - \vec\beta \cdot \frac{\vec\sigma}{2})\psi_j$$

$$= \psi_i^T c \psi_j$$

that is $\psi_i^T c \psi_j$ transforms like a Lorentz scalar

then $\mathcal{L}$ is Lorentz-invariant

(4) $$\mathcal{L}_{Dirac} = \bar\Psi i\gamma \cdot \partial \Psi - m\bar\Psi\Psi$$

$$= (\psi_1^\dagger, \psi_2^T c) i \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_1 \\ -c\psi_2^* \end{pmatrix}$$

$$- m (\psi_1^\dagger, \psi_2^T c) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ -c\psi_2^* \end{pmatrix}$$

$$= \psi_1^\dagger i\bar\sigma^\mu \partial_\mu \psi_1 + \psi_2^T c \, \sigma^\mu \partial_\mu(-c\psi_2^*)$$

$$- m\psi_1^\dagger(-c\psi_2^*) - m\psi_2^T c \psi_1$$

$$= \psi_1^\dagger i\bar\sigma^\mu \partial_\mu \psi_1 + (-c\psi_2^*)^T i(\sigma^\mu)^T(-\partial_\mu) c^T \psi_2 (-1)$$
$$- m\psi_2^T c\psi_1 + m\psi_1^\dagger c\psi_2^*$$

since $\quad c^T(\sigma^\mu)^T c^T = c(\sigma^\mu)^T c = c(1,\vec\sigma^T)c$
$$= (1,\vec\sigma)c^2 = -\bar\sigma^\mu$$

$$= \psi_1^\dagger i\bar\sigma^\mu\partial_\mu\psi_1 + \psi_2^\dagger i\bar\sigma^\mu\partial_\mu\psi_2$$
$$-m(\psi_2^T c\psi_1 - \psi_1^\dagger c\psi_2^*) \qquad \text{QED}$$

(b) The field equation for ψ_i is

$$i\bar\sigma^\mu\partial_\mu\psi_i + m_{ij}^* c\psi_j^* = 0$$

m is complex symmetric, so we can represent it as
$$m = U\bar m U^T$$
where $\bar m$ is complex diagonal and U is unitary. Alternately
$$m^* = V^\dagger \bar m^* V^* \qquad \text{where} \quad U^* = V^\dagger$$
Then let $\psi' = V\psi$, the equation becomes diagonal in i
$$i\bar\sigma^\mu\partial_\mu\psi_i' + \bar m_i^* c\psi_i^* = 0$$
Now look for a solution
$$\psi = \binom{a}{b}e^{-ip\cdot x} \qquad p = (E,0,0,p)$$
Better, since the lower component is the complex conjugate
$$\psi = \begin{pmatrix} ae^{-ip\cdot x} \\ be^{+ip\cdot x}\end{pmatrix}$$
$$\bar\sigma\cdot p = \begin{pmatrix} E-p & 0 \\ 0 & E+p\end{pmatrix} \qquad \text{so}$$

$$+(E-p)\,a = -\bar{m}^*(-b)^*$$
$$-(E+p)\,b = -\bar{m}^*\,a^*$$
$$(E^2-p^2)\,a = \bar{m}\cdot\bar{m}\,a$$
$$\left((E^2-p^2)-|\bar{m}|^2\right)a = 0$$

so each eigenvalue $\bar{m}_i$ of m generates a massive fermion of mass $m^2 = |\bar{m}|^2$.

(c) $$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + \psi^\dagger i\bar{\sigma}\cdot\partial\psi - \tfrac{1}{2}m(\psi^T c\psi + h.c.)$$
$$+ F^*F + mF\phi + mF^*\phi^*$$

In the 2^{nd} line, complete the square.

$$= (F^* + m\phi)(F + m\phi^*) - m^2|\phi|^2$$

Integral over F:

$$\int \mathcal{D}F\,\mathcal{D}F^*\; e^{i\int d^4x\,(F^*+m\phi)(F+m\phi^*)} = \text{const.}$$

we obtain

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \psi^\dagger i\bar{\sigma}^\mu\partial_\mu\psi$$
$$-m^2|\phi|^2 - \tfrac{1}{2}m(\psi^T c\psi + h.c.)$$

This gives one complex boson and one Majorana fermion with mass

$$m^2 =$$

(d) Similarly, ~~integrating~~ out FF^* in the Lagrangian (5)

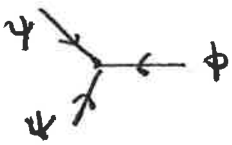
$$\mathcal{L} = \partial_\mu\phi^\dagger \partial^\mu \phi + \psi^\dagger i\bar{\sigma}\cdot\partial\psi - \eta_1^2 |\phi|^4 - \eta_2\left(\psi^T c\,\psi\,\phi - \phi^* \psi^\dagger c\,\psi^*\right)$$

the propagators are

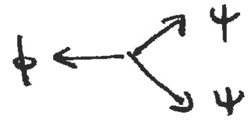
$\phi \xleftarrow{\quad} q$ $\quad \dfrac{i}{q^2}$ $\qquad \psi \xleftarrow{\quad} q$ $\quad \dfrac{i\sigma\cdot q}{q^2} = \langle \psi\,\psi^\dagger\rangle$

note order

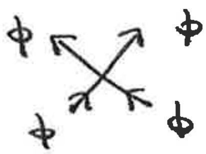
the vertices are (up to fermion minus signs)



$2i\eta_2 c$

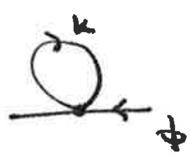


$+2i\eta_2 c$

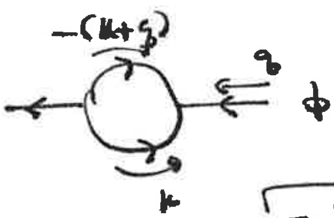


$-4i\eta_1^2$

the boson self-energy diagrams are



$$= -4i\eta_1^2 \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2} = 4\eta_1^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}$$



$$= (-2i\eta_2)\cdot\frac{1}{2}(2i\eta_2)$$

$$\psi^T c\,\psi\,\psi^\dagger c\,\psi^* \to (+1)\cdot \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[c\,\frac{i\sigma\cdot k}{k^2}\,c\left[\frac{i\sigma\cdot(-k-q)}{(k+q)^2}\right]^T\right]$$

no fermion interchange.

$$= 2\eta_2^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\frac{\sigma\cdot k}{k^2}\,c\,\frac{\sigma\cdot(k+q)}{(k+q)^2}\,c\right]$$

now $\quad c(\sigma^\mu)^T c = \bar\sigma^\mu c^2 = -\bar\sigma^\mu$

$$\mathrm{tr}\, \sigma^\mu \bar\sigma^\nu = 2 g^{\mu\nu}$$

$$= -4\eta_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{k\cdot(k+q)}{k^2 (k+q)^2}$$

The quadratic divergences are

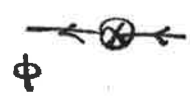
$$+ 4\eta_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} - 4\eta_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}$$

and they cancel if $\eta_1 = \eta_2$

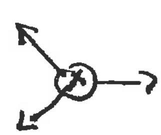
(e) The counterterm Lagrangian is

$$\Delta \mathcal{L} = \delta_\phi \partial^\mu \phi^* \partial_\mu \phi + \delta_\psi \psi^\dagger i \bar\sigma\cdot\partial \psi - \delta_{\eta_1^2} |\phi|^4$$
$$- \delta_{\eta_2} (\psi^T c \psi \phi - \phi^* \psi^\dagger c \psi^*) - \delta_m |\phi|^2$$

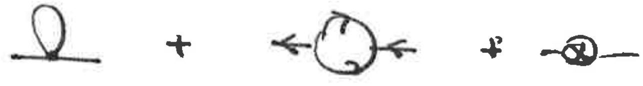
the counterterm vertices are

 $\quad -i\delta_m + i\delta_\phi q^2$ $\qquad$  $\quad i\delta_\psi \bar\sigma\cdot q$

 $\quad -4i\delta_{\eta_1^2}$

 $\quad -2i\delta_{\eta_2} c$ $\qquad$  $\quad +2i\delta_{\eta_2} c$

boson self-energy



we have already computed these diagrams. The counterterm

δm will cancel the pole at $d=2$. We are only interested now in the behavior near $d=4$:

$$= -4\eta_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{k\cdot(k+q)}{k^2(k+q)^2}$$

$$\mathbb{k} = k + xq \qquad k = \mathbb{k} - xq \qquad k+q = \mathbb{k} + (1-x)q$$

$$= -4\eta_2^2 \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4} \frac{(\mathbb{k}-xq)\cdot(\mathbb{k}+(1-x)q)}{[\mathbb{k}^2-\Delta]^2} \qquad \Delta = -x(1-x)q^2$$

$$= -4\eta_2^2 \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \left\{ -\frac{d}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} - x(1-x)q^2 \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \right\}$$

$$= -4\eta_2^2 \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left\{ -\frac{d}{2}\frac{(-x(1-x)q^2)}{1-d/2} - x(1-x)q^2 \right\}$$

$$= -4\eta_2^2 \int_0^1 dx \left[\frac{i}{(4\pi)^2} \log \Lambda^2/\Delta \right] \{-(2+1)\, x(1-x)q^2\}$$

$$= -4\eta_2^2 \left(\frac{i}{(4\pi)^2} \log \Lambda^2/\Delta \right) \left(\frac{-3}{6} \right) q^2$$

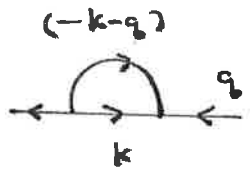
$$= +2\eta_2^2 \frac{i}{(4\pi)^2} (\log \Lambda^2/\Delta)\; i q^2$$

to cancel this

$$\delta_\phi = -2\eta_2^2 \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

fermion self-energy

$$\psi_c^{+}\,\psi^{+}\,\psi^{T} c\,\psi$$



$$= (2i\eta_2)(-2i\eta_2)\int\frac{d^4k}{(2\pi)^4}\; c\,\frac{i\left(\sigma\cdot(+k)\right)^T i}{k^2\,(k+q)^2}\,c\;(-1)$$

fermion interchange

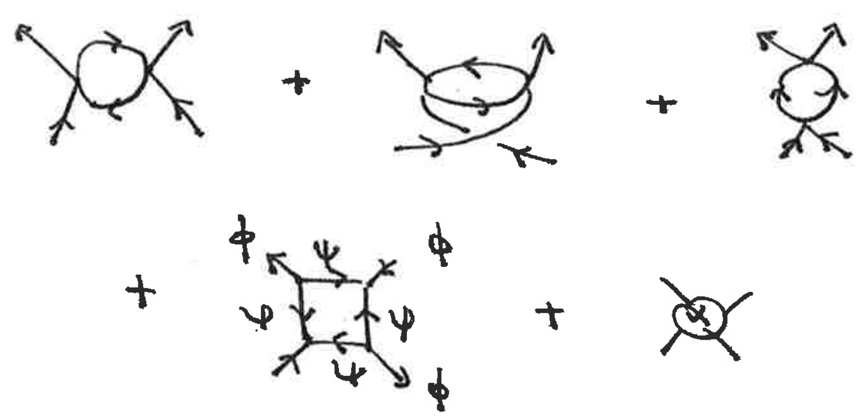
$$= +\,4\eta_2^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\; c\,\frac{[\sigma\cdot(k-xq)]^T}{(k^2-\Delta)^2}\,c$$

$$= -\,4\eta_2^2\int_0^1 dx\;\frac{i}{(4\pi)^2}\log(\Lambda^2/\Delta)\cdot\bar\sigma\cdot(-xq)$$

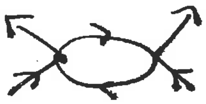
$$= +\,2\eta_2^2\,\frac{1}{(4\pi)^2}\log\Lambda^2/\Delta$$

$$\delta_\psi = -\,2\eta_2^2\,\frac{1}{(4\pi)^2}\log\Lambda^2/M^2$$

boson vertex:

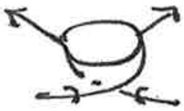


We can evaluate this diagram w/ external momenta = 0



$$= (-4i\eta_1^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2}\frac{i}{k^2}$$

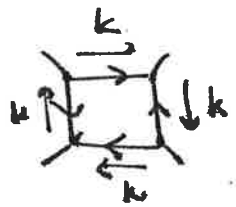
$$= 16\eta_1^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = 16\eta_1^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$



$$= (\text{same}) = 16\eta_1^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

$$= (-4i\eta_1^2)^2 \cdot \frac{1}{2} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2}\frac{i}{k^2}$$

$$= 8\eta_1^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$



$$\psi^T c\, \psi\, \psi^T c\, \psi^*\, \psi^T c\, \psi\, \psi^T c\, \psi^*$$

$$= (-2i\eta_2)^2 (2i\eta_2)^2 \cdot (-1) \quad \text{fermion interchange.}$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[c\, \frac{i\sigma\cdot k}{k^2}\, c\left(\frac{i\sigma\cdot(-k)}{k^2}\right)^T c \left(\frac{i\sigma\cdot k}{k^2}\right) c\left(\frac{i\sigma\cdot(-k)}{k^2}\right)\right]$$

$$= -16\eta_2^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^4} \operatorname{tr}\left[\sigma\cdot k\, \underbrace{c(\sigma\cdot k)^T c}_{-\bar\sigma\cdot k}\, \sigma\cdot k\, \underbrace{c(\sigma\cdot k)^T c}_{-\bar\sigma\cdot k}\right]$$

$$\sigma\cdot k\, \bar\sigma\cdot k = k^2$$

$$= -32\eta_2^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = -32\eta_2^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

in all

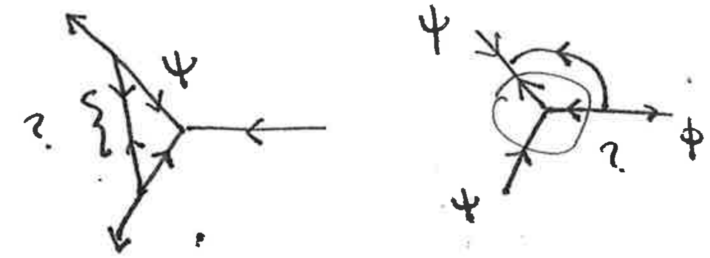
$$(2\cdot 16\eta_1^4 + 8\eta_1^4 - 32\eta_2^4) \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta - 4i\,\delta\eta_1^2 = \text{finite}$$

$$\delta\eta_1^2 = \left[2\eta_1^4 + 8(\eta_1^4 - \eta_2^4)\right] \frac{1}{(4\pi)^2} \lg \Lambda^2/M^2$$

fermion vertex

there is correction to the fermion vertex, since there is no way to connect the lines, preserving the direction of the arrows.

e.g.



$$\delta_{\eta_2} = 0$$

Now assemble the β-functions:

$$\beta_{\eta_1^2} = M\frac{\partial}{\partial M}\left[-\delta_{\eta_1^2} + \frac{4}{2}\delta_\phi \eta_1^2\right]$$

$$= M\frac{\partial}{\partial M}\left[-\frac{1}{(4\pi)^2}\lg \Lambda^2/M^2\right]\left[+\left(2\eta_1^4 + 8(\eta_1^4 \eta_2^4)\right) + 2\eta_1^2(+2\eta_2^2)\right]$$

$$= \frac{2}{(4\pi)^2}\left[2\eta_1^4 + 4\eta_1^2\eta_2^2 + 8(\eta_1^4 - \eta_2^4)\right]$$

$$= \frac{1}{4\pi^2}\left[\eta_1^4 + 2\eta_1^2\eta_2^2 + 4(\eta_1^4 - \eta_2^4)\right]$$

$$\beta_{\eta_2} = M\frac{\partial}{\partial M}\left[-\delta_{\eta_2} + \frac{3}{2}\delta_\psi \eta_2 + \frac{1}{2}\delta_\phi \eta_2\right]$$

$$= M\frac{\partial}{\partial M}\left[-\frac{1}{(4\pi)^2}\lg \Lambda^2/M^2\right]\left[-0 + 2\eta_2^3 + 1\cdot\eta_2^3\right]$$

$$= \frac{1}{(4\pi)^2}\, 2\cdot 3\,\eta_2^3$$

so

$$\beta_{\eta_1^2} = \frac{2}{8\pi^2}\left[3\eta_2^4 + 2\eta_2^2(\eta_1^2-\eta_2^2) + 10(\eta_1^4-\eta_2^4)\right]$$

$$\beta_{\eta_2} = \frac{1}{8\pi^2}\left[3\eta_2^3 \right]$$

then

$$\mu\frac{\partial}{\partial\mu}\eta_2^2 = 2\eta_2\beta_\eta = \frac{2}{8\pi^2}(3\eta_2^4)$$

so if $\eta_1 = \eta_2$, this relation is preserved by RG evolution

$$\mu\frac{\partial}{\partial\mu}(\eta_1^2-\eta_2^2) = \frac{2}{8\pi^2}(\eta_1^2-\eta_2^2)(2\eta_2^2 + 10(\eta_1^2+\eta_2^2))$$

$$= +\frac{2}{8\pi^2}(\eta_1^2-\eta_2^2)(22\eta_2^2) + \cdots$$

so $(\eta_1^2-\eta_2^2)$ increases to the UV

decreases to the IR

the condition $\eta_1^2 = \eta_2^2$ is stable as one evolves to the IR (as long as the mass of ϕ is subtracted)

(f) In supersymmetric Yang-Mills theory, the β function is

$$\beta = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \underbrace{\frac{2}{3}C_2(G)}\right]$$

the contribution of a chiral fermion is $\frac{1}{2}\cdot\frac{4}{3}$

$C(G) = C_2(G)$

or

$$\beta = -\frac{3g^3}{(4\pi)^2}C_2(G)$$

(g) The β function of the model (8) is

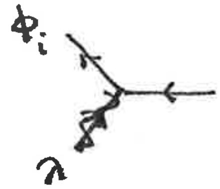
$$\beta = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - \frac{2}{3}C(R) - \frac{1}{3}C(R)\right]$$

$$= -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - C(R)\right]$$

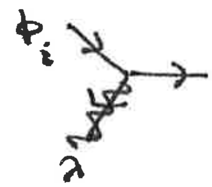
coupling to $\underline{3}$ Wess-Zumino multiplets in the adjoint representation gives $\underline{\beta = 0}$.

(h) For
$$\Delta\mathcal{L} = -\sqrt{2}\,g_1\left(\phi^\dagger \lambda^{Ta} c\, t^a_R \psi - \psi^\dagger c \lambda^{a*} t^a_R \phi\right) - \tfrac{1}{2} g_2 \left(\phi^\dagger t^a_R \phi\right)^2$$

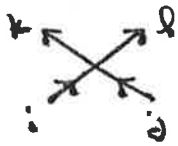
the Feynman rules are: (up to fermion minus signs)

ϕ_i  ψ_j $= -i\sqrt{2}\,g_1\, c(t^a_R)_{ij}$

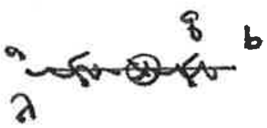
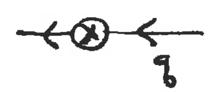
λ

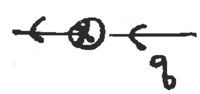
ϕ_i  ψ_j $= +i\sqrt{2}\,g_1\, c(t^a_R)_{ji}$

λ

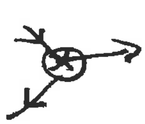
k, l, i, j  $= -i\, g_2^2\left[(t^a_R)_{ki}(t^a_R)_{lj} + (t^a_R)_{kj}(t^a_R)_{li}\right]$

the counterterm vertices are:

a λ  q b $\quad i\delta_\lambda\, \bar\sigma\cdot q\, \delta^{ab}$ $\qquad$ ϕ  q $\quad i\delta_\phi q^2 - i\delta_m$

ψ  q $\quad i\delta_\psi\, \bar\sigma\cdot q$

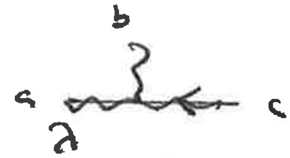
 $-i\sqrt{2}\,\delta_{g_1}\; c(t^a_R)_{ij}$

 $+i\sqrt{2}\,\delta_{g_1}\; c(t^a_R)_{ji}$

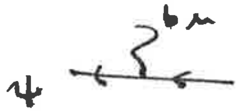
 $-i\delta_{g_2^2}\left[(t^a_R)_{ki}(t^a_R)_{lj} + (t^a_R)_{kj}(t^a_R)_{li}\right]$

I will work in Feynman gauge  $= \dfrac{-ig^{\mu\nu}}{q^2}$

We also need the gauge vertices

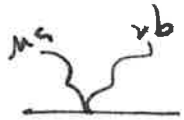


$$= ig\,(t^b_G)_{ac}\,\bar\sigma^\mu = -g f^{abc}\,\bar\sigma^\mu$$



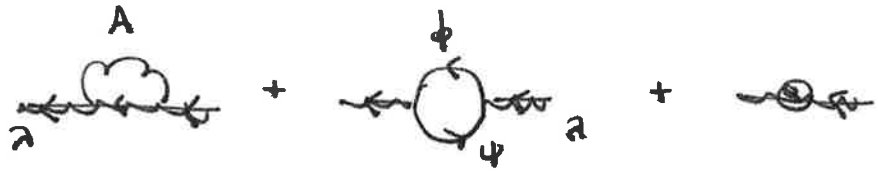
$$= ig\, t^b_R\, \bar\sigma^\mu$$

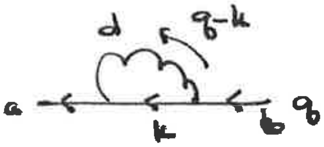
$$= ig(k+k')^\mu\, t^b_R$$



$$= ig^2 g^{\mu\nu} \{t^a_R, t^b_R\}$$

λ self-energy





$$= (ig)^2 (t^d_G\, t^d_G)_{ab}$$

$$\cdot \int \frac{d^4k}{(2\pi)^4}\, \bar\sigma^\mu \frac{i\sigma\cdot k}{k^2}\, \bar\sigma_\mu \frac{-i}{(k-q)^2}$$

$$= -g^2 C_2(G)\, \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{\bar\sigma^\mu\, \sigma\cdot k\, \bar\sigma_\mu}{k^2\,(k-q)^2}$$

$$= 2g^2 C_2(G)\, \delta^{ab} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\bar\sigma\cdot(\ell + xq)}{k^2\,(k-q)^2}$$

$$= 2g^2 C_2(G)\, \delta^{ab} \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}}\, \bar\sigma\cdot xq$$

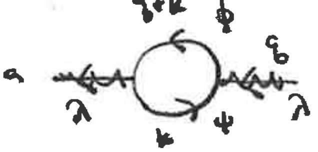
$$= g^2 C_2(G)\, \delta^{ab} \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta\;\; \bar\sigma\cdot q$$

$\ell = k - xq$

$k = \ell + xq$

$k - q = \ell - (1-x)q$

$\bar\sigma^\mu \sigma^\lambda \bar\sigma_\mu = -2\bar\sigma$



$= (-i\sqrt{2}g_1)(i\sqrt{2}g_1)\ \mathrm{tr}\ t_R^a t_R^b$

$\lambda^\dagger c\, \psi^* \psi^T c\, \psi$ $\qquad (-1)$

$\cdot \int \frac{d^4k}{(2\pi)^4}\ c \left(\frac{i\sigma\cdot k}{k^2}\right)^T c\ \frac{i}{(k+q)^2}$ $\qquad \ell = k + xq$, $\quad k = \ell - xq$

$$= 2g_1^2\, C(R)\, \delta^{ab} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\ \frac{-\bar\sigma\cdot(\ell - xq)}{[\ell^2-\Delta]^2}$$

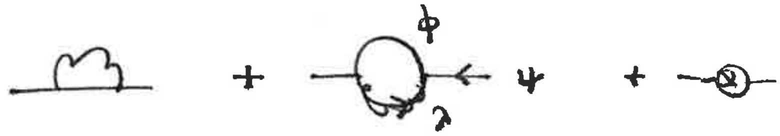
$$= 2g_1^2\, C(R)\, \delta^{ab}\ \frac{1}{2}\ \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta \cdot \bar\sigma\cdot q$$

$$= g_1^2\, C(R)\, \delta^{ab}\ \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta\ \ \bar\sigma\cdot q$$

so

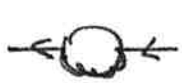
$$\delta_\lambda = -\left[g^2 C_2(G) + g_1^2 C(R)\right] \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

ψ self-energy



the first two diagrams are the same as those for λ.
except that the group thy factor is $t_R^a t_R^a = C_2(R)$

$= g^2\, C_2(R)\ \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta\ \ \bar\sigma\cdot q$

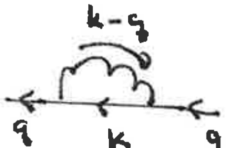


$= g_1^2\, C_2(R)\ \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta\ \ \bar\sigma\cdot q$

$$\delta_\psi = -\left[(g^2 + g_1^2)\, C_2(R)\right] \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

ϕ self-energy

 ,  have no terms $\propto q^2$



$$= (ig)^2\, t^a_R t^a_R \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2}\, (k+q)^\mu (k+q)_\mu\, \frac{-i}{(k-q)^2}$$

$$= -g^2 C_2(R) \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{(\ell + (1+x)q)^2}{[\ell^2 - \Delta]^2}$$

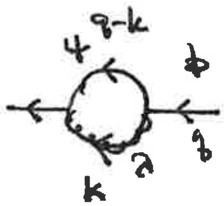
$\ell = k - xq$
$k = \ell + xq$
$k+q = \ell + (1+x)q$
$\Delta = -x(1-x)q^2$

$$= -g^2 C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \left\{ -\frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} + (1+x)^2 q^2 \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \right\}$$

$$= -g^2 C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \left\{ \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \right\} \left\{ -\frac{d}{2} \frac{(-x(1-x)q^2)}{1-\frac{d}{2}} + (1+x)^2 q^2 \right\}$$

$$= -g^2 C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \big(\underbrace{-2x(1-x) + 1 + 2x + x^2}_{1+3x^2}\big)\, q^2$$

$$= -2 g^2 C_2(R)\, \frac{i}{(4\pi)^2}\, \log \Lambda^2/\Delta \;\; q^2$$



$$(-i\sqrt{2}g_1)(i\sqrt{2}g_1)\; t^a_R t^a_R$$

$$\overbrace{\lambda^T c \psi \;\; \psi^T c \lambda^*} \longrightarrow (+1)$$

$$\cdot \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[c\, \frac{i\, \sigma\cdot(q-k)}{(q-k)^2}\, c \left(\frac{i\sigma\cdot k}{k^2}\right)^T \right]$$

$\ell = k - xq$
$k = \ell + xq$
$k - q = \ell - (1-x)q$

$$= 2 g_1^2 C_2(R) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\frac{\sigma\cdot(k-q)}{(k-q)^2}\, c \left(\frac{\sigma\cdot k}{k^2}\right)^T c \right]$$

$$= -2 g_1^2 C_2(R) \int \frac{d^4k}{(2\pi)^4}\, \frac{k(k-q)}{(k-q)^2 k^2} \cdot 2$$

$$= -4 g_1^2 C_2(R) \int dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\ell^2 - x(1-x)q^2}{[\ell^2 - \Delta]^2}$$

$$= -4g_1^2\, C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \left\{ -\frac{d}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} - x(1-x)q^2\, \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \right\}$$

$$= -4g_1^2\, C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left\{ -\frac{d}{2}\frac{(-x(1-x)q^2)}{(1-d/2)} - x(1-x)q^2 \right\}$$

$$= -4g_1^2\, C_2(R) \int_0^1 dx\, \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left\{ -3x(1-x)q^2 \right\}$$

$$= +2g_1^2\, C_2(R)\, \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta \quad q^2$$

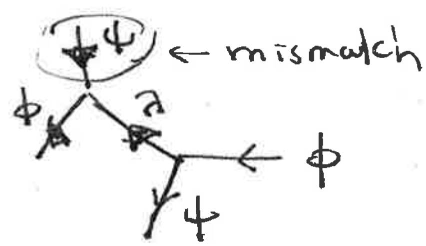
then

$$\delta_\phi = 2(g^2 - g_1^2)\, C_2(R)\, \frac{1}{(4\pi)^2} \log \Lambda^2/M^2$$

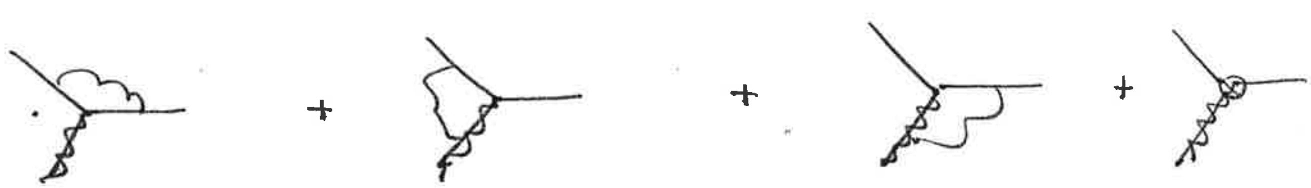
fermion vertex

As in part (e) there is no 1-loop fermionic correction to the fermion vertex

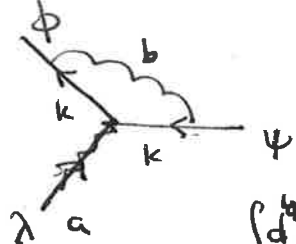
for example



so we have only



We may evaluate these diagrams in zero external momentum



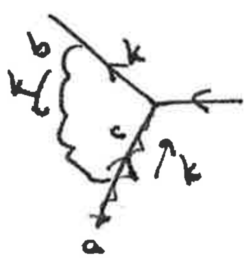
$$= (ig)^2\, t^b t^a t^b \,(-i\sqrt{2}\, g_1)$$

$$\int \frac{d^4k}{(2\pi)^4} \quad k^\mu \; \frac{i}{k^2} \; c \; \frac{i\bar\sigma\cdot k}{k^2} \; \bar\sigma^\mu \; \frac{-i}{k^2}$$

$$= -\sqrt{2}\, g_1 g^2 \,(t^b t^a t^b) \int \frac{d^4k}{(2\pi)^4}\, c\, \frac{\sigma\cdot k\,\bar\sigma\cdot k}{(k^2)^3}$$

$$\begin{aligned} t^b t^a t^b &= t^b t^b t^a + t^b [t^a, t^b] \\ &= C_2(R)\, t^a + t^b\, i f^{abc}\, t^c \\ &= C_2(R)\, t^a + \tfrac{1}{2}[t^b, t^c]\; i f^{abc} \\ &= C_2(R)\, t^a + \tfrac{1}{2} i f^{bcd}\, t^d\, i f^{abc} = \left(C_2(R) - \tfrac{1}{2} C_2(G)\right) t^a \end{aligned}$$

$$= -\sqrt{2}\, g_1 g^2 \left(C_2(R) - \tfrac{1}{2} C_2(G)\right) t^a \,\frac{i}{(4\pi)^2} \log \Lambda^2/\Delta$$



$$= (ig)\,(-i\sqrt{2}\, g_1)\; t^b\, (-g f^{cba})\, t^c$$

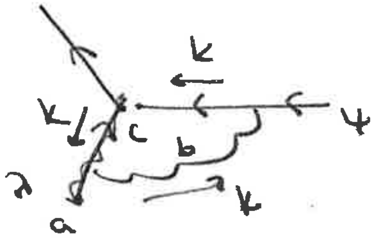
$$\lambda^* \bar\sigma \lambda \quad \lambda^T c\, \psi \qquad \cdot (+1)$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \quad k^\mu \; \frac{i}{k^2} \; (\bar\sigma^\mu)^T \left(\frac{i\sigma\cdot k}{k^2}\right)^T c \quad \frac{-i}{k^2}$$

$$= +i\sqrt{2}\, g_1 g^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2}\right)^3 c\;\sigma\cdot k\;\bar\sigma\cdot k \quad t^b t^c f^{abc}$$

$$= i\sqrt{2}\, g_1 g^2 \int \frac{dk}{(2\pi)} \,\frac{k^2}{(k^2)^3} \;\; \tfrac{1}{2}[t^b, t^c]\, f^{abc}$$

$$= -\sqrt{2}\, g_1 g^2 \;\frac{i}{(4\pi)^2} \log \Lambda^2/\Delta \;\; c\, t^a \;\; \tfrac{1}{2} C_2(G)$$



$$= (-i\sqrt{2}g_1)(ig)(-gf^{cba})\, t^c t^b$$

$$\lambda^\dagger\bar\sigma\lambda \quad \lambda^T c\,\psi \quad \psi^\dagger\sigma\psi \qquad (+1)$$

$$\cdot\int\frac{d^4k}{(2\pi)^4}\quad (\bar\sigma^\mu)^T\left(\frac{i\sigma\cdot(-k)}{k^2}\right)^T c\;\frac{i\sigma\cdot k}{k^2}\,\bar\sigma_\mu\quad\frac{-i}{k^2}$$

$$= i\sqrt{2}g_1 g^2\,(f^{acb}t^c t^b)\int\frac{dk}{(2\pi)}\; c\,\sigma^\mu\,\bar\sigma\cdot k\,\sigma\cdot k\,\bar\sigma_\mu\;\frac{1}{(k^2)^3}$$

$$= i\sqrt{2}g_1 g^2\left(\tfrac{1}{2}i f^{acb}f^{cbd}t^d\right)\int\frac{d^4k}{(2\pi)^4}\; c\;\underbrace{\sigma^\mu\bar\sigma_\mu}_{4}\;\frac{1}{(k^2)^2}$$

$$= -\sqrt{2}g_1 g^2\;\frac{4}{2}C_2(G)\,t^a c\;\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta$$

in all

$$(-i\sqrt{2}\,ct^a)\cdot\left[\,g_1g^2\left((C_2(R)-\tfrac{1}{2}C_2(G)) + \tfrac{1}{2}C_2(G) + 2\,C_2(G)\right)\right]$$

$$\cdot\frac{1}{(4\pi)^2}\log\Lambda^2/\Delta \;+\; (-i\sqrt{2}\,ct^a\,\delta_{g_1}) \;=\; \text{finite}$$

$$\delta_{g_1} = -\;g_1g^2\,(C_2(R)+2C_2(G))\;\frac{1}{(4\pi)^2}\log\Lambda^2/M^2$$

Now we can construct β_{g_1}. I claim ~~that~~, for $g_1=g$,

$$\beta_{g_1}(g,g) = \beta(g) \text{ on p. 12}$$

For this to happen, all factors of $C_2(R)$ must cancel.

$$\beta_{g_1} = M\frac{\partial}{\partial M}\left[-\delta_{g_1} + \frac{g_1}{2}(\delta_2 + \delta_\psi + \delta_\phi)\right]$$

$$= M\frac{\partial}{\partial M}\left(\frac{1}{(4\pi)^2}\lg \Lambda^2/M^2\right)$$

$$\cdot\Big[\, g_1 g^2 (C_2(R) + 2C_2(G))$$

$$+ \left(-\tfrac{1}{2} g_1 g^2 C_2(G) - \tfrac{1}{2} g_1^3 C(R)\right)$$

$$+ \left(-\tfrac{1}{2} g_1(g^2+g_1^2) C_2(R)\right)$$

$$+ \tfrac{1}{2}\cdot 2g_1\cdot(g^2-g_1^2) C_2(R)\Big]$$

$$= \frac{1}{(4\pi)^2}(-2)\Big[\tfrac{3}{2} C_2(G) g_1 g^2 - \tfrac{1}{2} g_1^3 C(R)$$

$$+ \left(\tfrac{3}{2} g_1 g^2 - \tfrac{3}{2} g_1^3\right) C_2(R)\Big]$$

$$\beta_{g_1} = -\frac{1}{(4\pi)^2}\Big[3 C_2(G) g_1 g^2 - C(R) g_1^3$$

$$+ 3(g_1 g^2 - g_1^3) C_2(R)\Big]$$

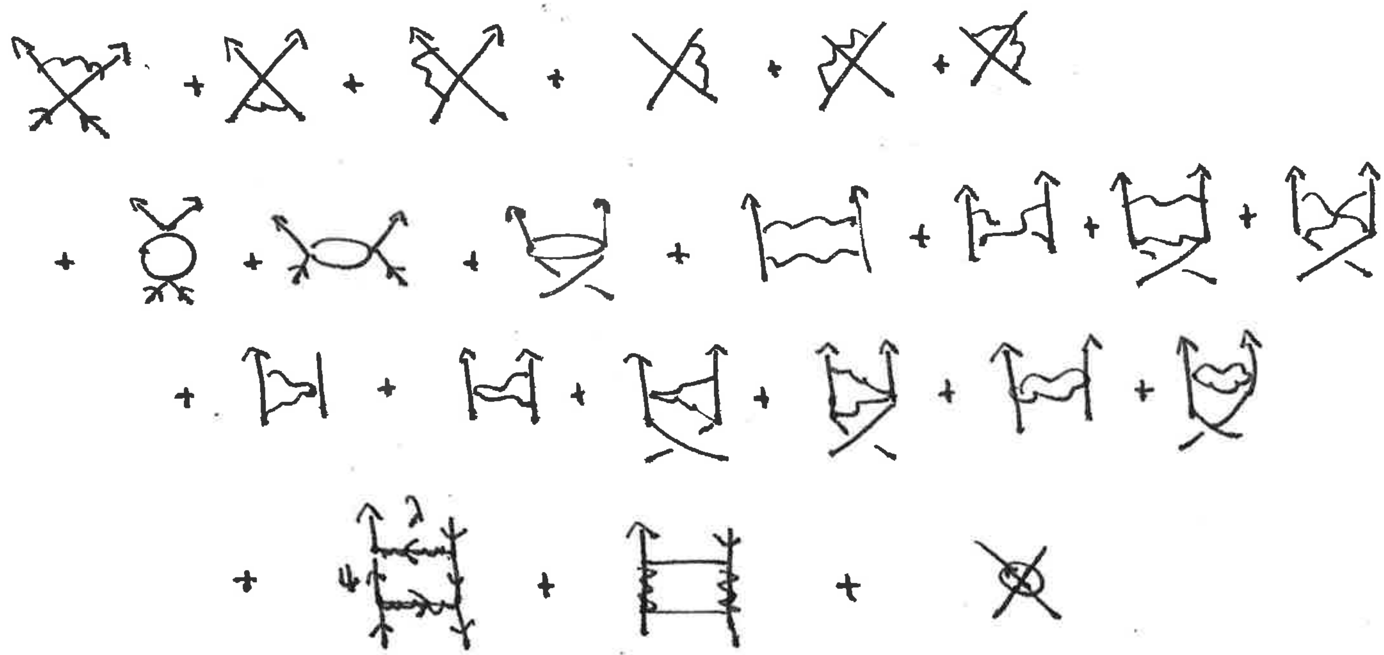
for $g_1 = g$

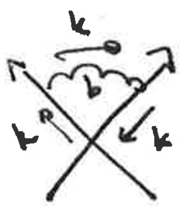
$$\beta_{g_1} = -\frac{g^3}{(4\pi)^2}\left[3C_2(G) - C(R)\right]$$

as required.

bosonic vertex

there are a very large number of diagrams. However, all can be computed at zero external momenta:

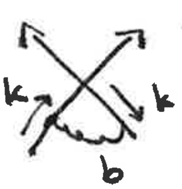




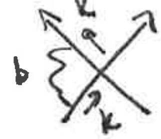
$$= (ig)^2(-ig_2^2)\int\frac{d^4k}{(2\pi)^4}\,k^\mu(-k_\mu)\,\frac{i}{k^2}\,\frac{i}{k^2}\,\frac{-i}{k^2}$$

$$\cdot\left[(t^bt^a)_{ki}(t^bt^a)_{lj}+(t^bt^a)_{kj}(t^bt^a)_{li}\right]$$

$$= +g^2g_2^2\,\frac{i}{(4\pi)^2}\lg\Lambda^2/\Delta\quad\left[(t^bt^a)_{ki}(t^bt^a)_{lj}+(t^bt^a)_{kj}(t^bt^a)_{li}\right]$$



$$= g^2g_2^2\,\frac{i}{(4\pi)^2}\lg\Lambda^2/\Delta\quad\left[(t^at^b)_{ki}(t^at^b)_{lj}+(t^at^b)_{kj}(t^bt^a)_{li}\right]$$



$$= (ig)^2(-ig_2^2)\int\frac{d^4k}{(2\pi)^4}\, k^\mu k_\mu \frac{i}{k^2}\frac{i}{k^2}\frac{-i}{k^2}$$

$$[\,(t^bt^at^b)_{ki}(t^a)_{\ell j} + (t^bt^a)_{kj}(t^at^b)_{\ell i}]$$

$$= -g^2g_2^2\,\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta\,\left[(t^bt^at^b)_{ki}(t^a)_{\ell j} + (t^bt^a)_{kj}(t^at^b)_{\ell i}\right]$$

Similarly

$$= -g^2g_2^2\,\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta\,\left[(t^a)_{ki}(t^bt^at^b)_{\ell j} + (t^at^b)_{kj}(t^bt^a)_{\ell i}\right]$$

$$= -g^2g_2^2\,\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta\,\left[(t^bt^a)_{ki}(t^at^b)_{\ell j} + (t^bt^at^b)_{kj}(t^a)_{\ell i}\right]$$

$$= -g^2g_2^2\,\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta\,\left[(t^at^b)_{ki}(t^bt^a)_{\ell j} + (t^a)_{kj}(t^bt^at^b)_{\ell i}\right]$$

now $\quad t^bt^at^b = [C_2(R) - \frac{1}{2}C_2(G)]\,t^a$

and we can use this to simplify 4 terms. The other 8 terms organize into

$$([t^a,t^b])_{ki}([t^a,t^b])_{\ell j} + ([t^a,t^b])_{kj}([t^a,t^b])_{\ell i}$$

$$= if^{abc}(t^c)_{ki}\; if^{abd}(t^d)_{\ell j} + (i\leftrightarrow j)$$

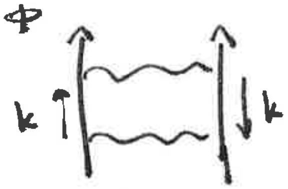
$$= -\,C_2(G)\,(t^a)_{ki}(t^a)_{\ell j} + (i\leftrightarrow j)$$

in all

$$(6\text{ diagrams}) = -g^2g_2^2\,\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta$$

$$\cdot\left\{2\,[C_2(R)-\tfrac{1}{2}C_2(G)] + C_2(G)\right\}\left((t^a)_{ki}(t^a)_{\ell j} + i\leftrightarrow j\right).$$

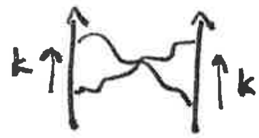
Next:



$$= (ig)^4 \, (t^a t^b)_{ki} (t^a t^b)_{lj} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{k^2} \frac{-i}{k^2} \frac{-i}{k^2}$$

$$\cdot \, k^\mu k^\nu (-k^\mu)(-k^\nu)$$

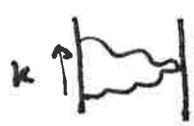
$$= g^4 \, (t^a t^b)_{ki} (t^a t^b)_{lj} \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$



$$= (ig)^4 (t^a t^b)_{ki} (t^b t^a)_{lj} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{k^2} \frac{-i}{k^2} \frac{-i}{k^2}$$

$$\cdot \, k^\mu k^\nu k^\mu k^\nu$$

$$= g^4 (t^a t^b)_{ki} (t^b t^a)_{lj} \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$



$$= (ig)^2 (ig^2) \, (t^a t^b)_{ki} (\{t^a, t^b\})_{lj} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{-i}{k^2} \frac{-i}{k^2}$$

$$\cdot \, k^\mu g_{\mu\nu} k^\nu$$

$$= -g^4 (t^a t^b)_{ki} (t^a t^b + t^b t^a)_{lj} \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

$=$ 

$$= \frac{1}{2} (ig^2)^2 \, (t^a t^b + t^b t^a)_{ki} \, (t^a t^b + t^b t^a)_{lj} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{k^2} g^{\mu}$$

$$= +2g^4 (\{t^a, t^b\})_{ki} (\{t^a, t^b\})_{lj} \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

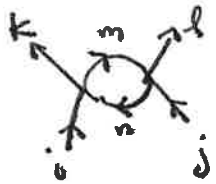
$\approx$ all

$$= g^4 \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta$$

$$\left\{ \frac{1}{2} (\{t^a,t^b\})_{ki} (\{t^a,t^b\})_{lj} - (\{t^a,t^b\})_{ki} (\{t^a,t^b\})_{lj} + 2 (\{t^a,t^b\})_{ki} (\{t^a,t^b\})_{lj} \right\}$$

and we must add → $\quad + (i \leftrightarrow j)$

Next



$$= (-i g_2^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{k^2}$$

$$\left\{ (t^a)_{ki}\, \mathrm{tr}(t^a t^b)\, t^b_{lj} + (t^a t^b t^a)_{ki} (t^b)_{lj} + (t^a)_{ki} (t^b t^a t^b)_{lj} + (t^a t^b)_{kj} (t^b t^a)_{li} \right\}$$

$$= g_2^4 \frac{i}{(4\pi)^2} \log \Lambda^2/\Delta$$

$$\cdot \left\{ (t^a)_{ki} (t^a)_{lj} \left[C(R) + 2 C_2(R) - C_2(G) \right] + (t^a t^b)_{kj} (t^b t^a)_{li} \right\}$$

$$= g_2^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

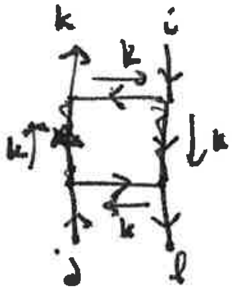
$$\{ (t^a)_{kj}(t^a)_{li} \left[d(R) + 2C_2(R) - C_2(G) \right] + (t^a t^b)_{ki}(t^b t^a)_{lj} \}$$

$$= \frac{1}{2} (-i g_2^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{k^2}$$

$$\cdot \{ (t^a t^b)_{ki}(t^a t^b)_{lj} \cdot 2 + (t^a t^b)_{kj}(t^a t^b)_{li} \cdot 2 \}$$

$$= + g_2^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

$$\{ (t^a t^b)_{ki}(t^a t^b)_{lj} + (t^a t^b)_{kj}(t^a t^b)_{li} \}$$



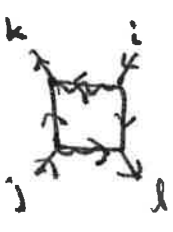
$$= (-i\sqrt{2} g_1)^2 (+i\sqrt{2} g_1)^2 \quad (t^a t^b)_{ki}(t^b t^a)_{lj}$$

$$\lambda^T c \psi \ \psi^+ c \lambda^* \ \lambda^T c \psi \ \psi^+ c \lambda^* \quad \cdot (-1)$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \mathrm{tr} \left(\frac{i\sigma\cdot(-k)}{k^2} \right) c \left(\frac{i\bar\sigma\cdot k}{k^2} \right)^T c \left(\frac{i\sigma\cdot(-k)}{k^2} \right) c \left(\frac{i\bar\sigma\cdot k}{k^2} \right)^T$$

$$= -4 g_1^4 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr} \frac{(\sigma\cdot k \ \bar\sigma\cdot k)^2}{(k^2)^4} \quad (t^a t^b)_{ki}(t^b t^a)_{lj}$$

$$= -8 g_1^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta \quad (t^a t^b)_{ki}(t^a t^b)_{lj}$$



$$= -8 g_1^4 \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta \quad (t^a t^b)_{kj}(t^b t^a)_{li}$$

the sum of all diagrams on p. 21 is

$$\left(\frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta \right)$$

$$\{ [(t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j)] \; [-2 g^2 g_2^2 \, C_2(R)]$$

$$+ [(\{t^a, t^b\})_{ki} (\{t^a, t^b\})_{\ell j} + (i \leftrightarrow j)] \; [\tfrac{3}{2} g^4]$$

$$+ [(t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j)] \; g_2^4 \, [C(R) + 2C_2(R) - C_2(G)]$$

$$+ [(t^a t^b)_{ki}(t^b t^a)_{\ell j} + (i \leftrightarrow j)] \; g_2^4$$

$$(t^a t^b) = (\tfrac{1}{2}\{t^a, t^b\} + \tfrac{1}{2}[t^a, t^b])$$

$$+ [(t^a t^b)_{ki}(t^a t^b)_{\ell j} + (i \leftrightarrow j)] \; g_2^4$$

$$+ [(t^a t^b)_{ki}(t^b t^a)_{\ell j} + (i \leftrightarrow j)] \; (-4 g_1^4) \}$$

$$= \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

$$\{ (t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j) \} [C_2(R) [2 g_2^4 - 2 g^2 g_2^2] + C(R) g_2^4 - C_2(G) g_2^4]$$

$$+ [(\{t^a, t^b\})_{ki}(\{t^a, t^b\})_{\ell j} + (i \leftrightarrow j)] [\tfrac{3}{2} g^4 + \tfrac{2}{4} g_2^4 - 2 g_1^4]$$

$$+ [([t^a, t^b])_{ki}([t^a, t^b])_{\ell j} + (i \leftrightarrow j)] [-\tfrac{1}{4} g_2^4 + \tfrac{1}{4} g_2^4 - 2 g_1^4]$$

$$= \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta \Big[\{ (t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j) \} [2C_2(R)(g_2^4 - g^2 g_2^2) + C(R) g_2^4 - C_2(G) g_2^4]$$

$$+ (\{t^a, t^b\})_{k\ell}(\{t^a, t^b\})_{\ell j} + (i \leftrightarrow j) [\tfrac{3}{2} g^4 + \tfrac{1}{2} g_2^4 - 2 g_1^4]$$

$$+ (i f^{abc} i f^{abd} [(t^c)_{ki}(t^d)_{\ell j} + (i \leftrightarrow j)] (-2 g_1^4) \Big]$$

$$= \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta$$

$$\Big[(t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j) \Big]$$

$$\Big[C_2(R)\,[2g_2^4 - 2g^2 g_2^2] - C_2(G)\,[g_2^4 + 2g_1^4] + C(R)\, g_2^4 \Big]$$

$$+ \Big[(\{t^a, t^b\})_{ki} (\{t^a, t^b\})_{\ell j} + (i \leftrightarrow j) \Big] \Big[\tfrac{3}{2} g^4 + \tfrac{1}{2} g_2^4 - 2 g_1^4 \Big] \Big]$$

If we set $g = g_1 = g_2$ this becomes

$$= \frac{i}{(4\pi)^2} \lg \Lambda^2/\Delta \quad \Big[(t^a)_{ki}(t^a)_{\ell j} + (i \leftrightarrow j) \Big]$$

$$\Big[-3 g^4 C_2(G) + g^4 C(R) \Big]$$

the $C_2(R)$ terms cancel out, and the new structure

$$(\{t^a, t^b\})_{ki} (\{t^a, t^b\})_{\ell j} + (i \leftrightarrow j)$$

which seems to be allowed by all symmetries, cancels out

Supersymmetry is magic!

Also at $g = g_1$ $\delta\phi = 0$ (see p. 17)

so we only need

$$\delta_{g_2^2} = \frac{g^4}{(4\pi)^2} \left(\lg \Lambda^2/m^2 \right) \left(-3\, C_2(G) + C(R) \right)$$

then

$$\beta_{g_2^2} = M\frac{\partial}{\partial M}\left[-\delta_{g_2^2} + \frac{4}{2}\delta_\phi\right]$$

$$= 2g\cdot\left(-\frac{g^3}{(4\pi)^2}\left[3C_2(G) - C(R)\right]\right)$$

To preserve $g = g_2$ we need

$$\beta_{g_2^2} = 2g\,\beta_g$$

and this relation is satisfied.

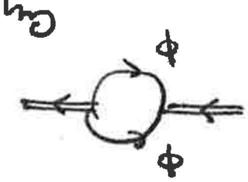
(i) [Please see the better formulation of this problem on the revised exam now posted.]

Go back to the Lagrangian (5) and treat F as a field. Its propagator is

$$\overline{F\,F^*} = i$$  F

It has a δ_F counterterm. 

given by

 +  $= 0$

$$(+2i\eta_1)^2\cdot\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{k^2} + i\delta_F = 0$$

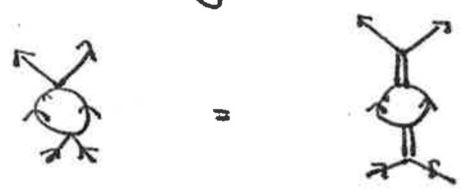
$$2\eta_1^2\frac{i}{(4\pi)^2}\log\Lambda^2/\Delta + i\delta_F = 0$$

so $$\delta_F = -\frac{2\eta_1^2}{(4\pi)^2}\log\Lambda^2/M^2$$

and

$$\delta_\phi = \delta_\psi = \delta_F \quad \text{for} \quad \eta_1 = \eta_2$$

The δ_F accounts the diagram



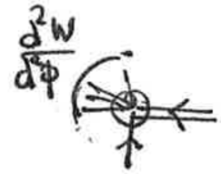
on p. 9

The other diagrams have a nice cancellation

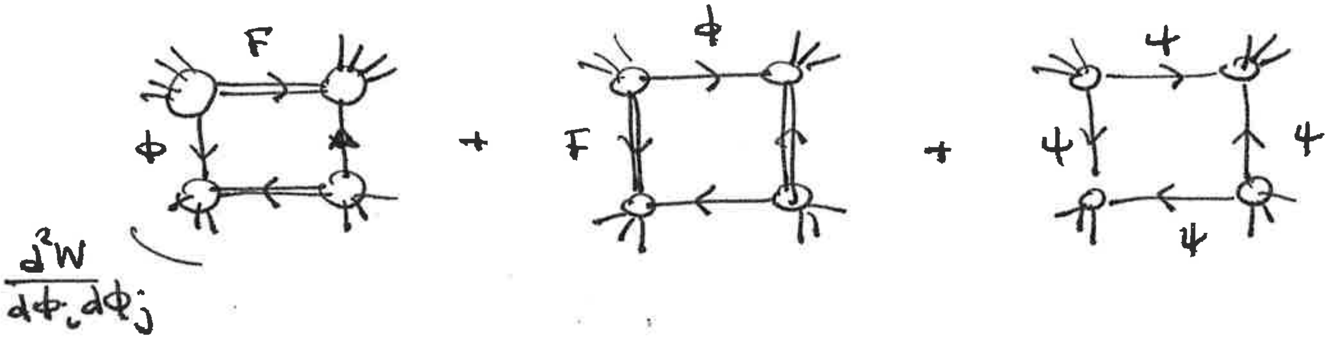
For the Lagrangian (8) this cancellation continues to hold

$$F_i \frac{dW}{d\phi_i} \to F_i \phi_j \frac{d^2W}{d\phi_i d\phi_j}$$

i.e.



and, with the same way,



$$= 0$$

Then: $W(\phi)$ is renormalized only by rescaling of fields. A term $A_{ij\ldots}\phi_i\phi_j\cdots$ omitted from $W(\phi)$ cannot be generated in perturbation theory.

this result, called "non-renormalization of the superpotential" is actually true to all orders in perturbation theory. It is another magic property of supersymmetry.