Physics 331 - Final Exam

due Friday, March 20, 2020, at noon

This is a take-home final exam. The rules for the exam are: (1) Do not collaborate with other students in the course or seek other human resources; (2) You may use any references that you find, but please do not actively search the web for the solutions to these problems. Both problems can be solved by just thinking about them using the methods that we have discussed in this course. If you find a very useful reference, though, please cite it in your solution.

The exam has 2 problems, worth 50 points each. Partial credit will be given. If you make progress, please put it down on paper.

If you find apparent typos on this exam, or if you have any other questions, please notify me immediately at mpeskin@slac.stanford.edu.

Turn in the exam to same box in Hewlett that you have used to hand in your homework. Please include with your exam a signed statement that you acknowledge and accept the Stanford Honor Code (e.g., tear off the cover of a Stanford blue book, sign the statement, and attach this to your solution). If you are unable to go to Hewlett, please (a) before Thursday, email both Chao Wang and me that you will turn in your exam electronically, and then (b) on Friday before noon PDT, scan your solution and email it to me.

1. A heavy neutral Higgs boson might be produced at the Large Hadron Collider through the vertex

$$\Delta\mathcal{L} = Y H \bar{b} b \ , \tag{1}$$

where Y is the coupling constant. The use of this vertex requires that there are b quarks in the proton. Actually, the low-momentum wavefunction of the proton does not contain many b quarks and antiquarks. The actual probability of finding a b in the proton at low momentum is proportional to $m_p^2/2m_b^2 \sim 1\%$. However, in a high-momentum transfer reaction, b quarks and antiquarks can arise through gluon splitting. In this problem, consider, for definiteness, a search for a heavy Higgs boson H of mass $m_H = 500$ GeV.

   (a) The H boson can be produced in the reaction $g + g \to H + b + \bar{b}$. Draw the Feynman diagrams contributing this reaction at the leading order in α_s.

   (b) It is somewhat complex to compute the complete cross section for this process, and this is not necessary to get a first approximation to the cross section. The total cross section is dominated by a term of order

$$\sigma \sim Y^2 \left(\frac{\alpha_s}{\pi}\right)^2 \log^2 \frac{m_H^2}{m_b^2} \ . \tag{2}$$

   Draw the (1) Feynman diagram that produces a term of this form.

(c) An ingredient in the cross section calculation for $g+g \to H+b+\overline{b}$ is the total cross section for $b+\overline{b} \to H$. Compute this cross section in the limit in which the width of the H is small. Note that this cross section is proportional to $(2\pi)\delta(s - m_H^2)$ in this limit.

(d) Using the idea of collinear splitting, compute the cross section for $g+g \to H+b+\overline{b}$ in the approximation that gives the log-enhanced contribution in eq. (2).

(e) Evaluate this cross section numerically for the following parameters: $Y = 1$, $E_{CM}(gg) = 1.4$ TeV. The former number is approximately the top quark Yukawa coupling. The latter number makes sense because we would be thinking about collisions at the LHC and the hardest gluons in the proton typically carry 10% of the total energy-momentum of the proton.

(f) Depending on the decay mode of H, the process $g+g \to H+b+\overline{b}$ might not be observable at the LHC in its most general kinematics. In particular, if the b and $\overline{b}$ are produced at angles close to the beam direction, they may not be recognized as products of the H production reaction rather than just quarks produced in other ways when the incoming protons are disrupted. To enhance the visibility of H production, we can insist that either the b or the $\overline{b}$ is produced with large momentum transverse to the beam direction. Then this particle can be recognized as a b-jet that is part of the H production process.

To analyze this strategy, we will need to study the reaction $bg \to Hb$. Draw the Feynman diagrams for this process.

(g) Compute the cross section, differential in the transverse momentum of the final b quarks,

$$\frac{d\sigma}{dp_T}(gb \to Hb) \tag{3}$$

(h) Assembling the pieces, compute the terms in the cross section

$$\frac{d\sigma}{dp_T}(gg \to Hb\overline{b}) \tag{4}$$

that are enhanced by one power of $\log m_H^2/m_b^2$.

(i) Plot the cross section in part (h) as a function of p_T. To evaluate this cross section numerically, estimate the bg CM energy as 0.99 TeV (ie., assume that the b quark carries exactly half the momentum of the parent gluon), rather than integrating over the b momentum fraction. This is not a good approximation, but it is OK for exam purposes.

(j) If we insist that the observed b quark jet has $p_T > 30$ GeV, what fraction of the total $gg \to Hb\overline{b}$ cross section would we be observing? This is the price of insisting on a visible b jet.

2. What if there existed a heavy multiplet of scalar fields in the isospin I representation of the electroweak $SU(2)$ gauge symmetry? We can see that this set of scalar fields has a potentially observable effect on the masses of the W and Z bosons. It is possible

to do these calculations for general I, but, if you get stuck, do them at least for the case $I = 1$.

Let φ be a vector of complex-valued scalar fields of dimension $(2I + 1)$, transforming under the isospin I representation of $SU(2)$ and with hypercharge Y under the electroweak $U(1)$. The covariant derivative of the electroweak theory is

$$D_\mu = \partial_\mu - ieA_\mu Q - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{c_w} Z_\mu Q_Z \ , \tag{5}$$

where

$$Q = T^3 + Y \ , \qquad Q_Z = T^3 - s_w^2 Q \ , \qquad T^\pm = (T^1 \pm iT^2) \tag{6}$$

and I abbreviate $s_w = \sin\theta_w$, $c_w = \cos\theta_w$.

(a) In leading order, the predictions of electroweak theory depend on the parameters g, g', and v. In class, we saw that the observable quantities of the electroweak theory are given at the leading order by the relations

$$\begin{aligned} \frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2} \quad &, \qquad e = \frac{gg'}{(g^2 + g'^2)^{1/2}} = gs_w \ , \\ m_W^2 = \frac{g^2 v^2}{4} \quad &, \qquad m_Z = \frac{(g^2 + g'^2)v^2}{4} \end{aligned} \tag{7}$$

In this problem, I suggest taking g, v, s_w as the basic bare Lagrangian parameters. Rearrange the equations (7) to write expressions for these variables, valid at the leading order, in terms of G_F, α, m_W. (In this problem, set $\alpha = 1/129$ and ignore the running of α.) Thus we have, for example,

$$v = 1/(\sqrt{2}G_F)^{1/2} + \cdots \tag{8}$$

where the omited terms are due to loop corrections.

(b) Compute the electromagnetic vacuum polarization diagram for the multiplet φ using dimensional regularization. Assume that the masses of all bosons in the φ multiplet have the same value M.

(c) Carry out the Feynman parameter integration in part (b) assuming $M^2 \gg q^2$, keeping terms up to and including order q^2/M^2.

(d) Including the result of part (b) will modify the equation for α in terms of g, s_w. Alternatively, it will modify the expressions for g, s_w in terms of observables. It will be convenient to write the first-order corrections to g and s_w in terms of

$$\frac{\Delta\alpha}{\alpha} \equiv \frac{\alpha(\text{1-loop}) - \alpha(\text{leading order})}{\alpha(\text{leading order})} \ . \tag{9}$$

Write the corrected form of the equations for g and s_w and show that the corrections are ultraviolet-divergent.

(e) Compute the vacuum polarization diagrams for W and Z using dimensional regularization. Show that these satisfy $q_\mu \Pi^{\mu\nu}(q) = 0$ under the assumption that all φ particles have the same mass M.

(f) Similarly to part (d), compute

$$\frac{\Delta G_F}{G_F} \quad \text{and} \quad \frac{\Delta m_W^2}{m_W^2} \tag{10}$$

(g) Construct the complete equations for v, g, s_w in terms of the observables G_F, m_W^2, α including all 1-loop effects of the multiplet φ. Show that these relations are all ultraviolet-divergent.

(h) Compute

$$\frac{\Delta m_Z^2}{m_Z^2} \ . \tag{11}$$

Now we can eliminate the bare parameters and construct an equation for m_Z^2 in terms of G_F, m_W^2, and α. Work out this equation to first order in φ loop corrections. Show that the ultraviolet infinities cancel, resulting in a concrete prediction for the mass shift of the Z from the lowest-order relation.

(i) Show that the terms in $\log M$ cancel, so that the predicted mass shift is of order $1/M^2$.

(j) The W mass is known to about 15 MeV, and the other parameters are known more accurately, so a deviation of the difference $m_Z - m_W$ of more than 50 MeV would signal a large deviation from the Standard Model. Write this condition as an lower bound on M, and work out this bound, which depends on I, Y. Work out the bound numerically for $I = 1$, $Y = 0$ and $I = 1$, $Y = 1$.

(k) We can split the masses of the φ bosons by adding an $SU(2) \times U(1)$-invariant interaction

$$\Delta\mathcal{L} = -\gamma \ (\varphi^\dagger T^a \varphi) \ (\Phi^\dagger \frac{\sigma^a}{2} \Phi) \ , \tag{12}$$

where Φ is the Standard Model Higgs field. What is the spectrum of φ masses when the Higgs acquires its expectation value v?

(l) Recompute the photon, W, and Z vacuum polarizations in this model, keeping terms up to and including linear order in γ. Notice that the W and Z vacuum polarizations are no longer satisfy $q_\mu \Pi^{\mu\nu}(q) = 0$. Fortunately, they are still free of quadratic divergences. Drop the terms proportional to $q^\mu q^\nu$ (Why is this OK?) and work with the terms proportional to $g^{\mu\nu}$.

(m) Compute the term in $m_Z - m_W$ proportional to γ.