Physics 331 - Final Exam

due Friday, March 22, 2019, at noon

This is a take-home final exam. The rules for the exam are: (1) Please do not collaborate with other students in the course or seek other human resources; (2) You may use any references that you find, but please do not actively search the web for the solutions to these problems. Both problems can be solved by just thinking about them using the methods that we have discussed in this course. If you find a very useful reference, though, please cite it in your solution.

The exam has 2 problems, worth 50 points each. Partial credit will be given. If you make progress, please put it down on paper.

If you find apparent typos on this exam, or if you have any other questions, please notify me immediately at mpeskin@slac.stanford.edu.

Please include with your exam a signed statement that you acknowledge and accept the Stanford Honor Code (e.g., tear off the cover of a Stanford blue book, sign the statement, and attach this to your solution). Turn in the exam to same box in Hewlett that you have used to hand in your homework.

1. A nonrelativistic quark-antiquark bound state with both the quark and antiquark spins up along the $\hat{3}$ direction can be represented by the quantum state

$$|B\rangle = \sqrt{2M} \int \frac{d^3k}{(2\pi)^3}\ \psi(k)\ \frac{1}{(\sqrt{2m})^2}\ |q(\uparrow)\overline{q}(\uparrow)\rangle\ \ , \qquad (1)$$

In this equation, m is the quark mass, M is the bound state mass, $\psi(k)$ is the Fourier transform of the bound state wavefunction, and the factors $\sqrt{2m}$ correct the relativistic normalization of the quark and antiquark states to the relativistic normalization of the bound state. Approximate $M = 2m$. Let the amplitude for a quark and antiquark with definite 2-component spinors ξ and $\overline{\xi}$ to annihilate to some final state be written

$$i\mathcal{M}(q\overline{q} \to f) = i\overline{\xi}^{\dagger}\Gamma(f)\xi\ . \qquad (2)$$

Remembering that the spinor in the antiquark spin is the reverse of the spin, and summing over spin configurations to obtain definite total spin S, the decay matrix element for an $S = 0$ bound state is

$$\mathcal{M}(B \to f) = \sqrt{\frac{2}{M}} \int \frac{d^3k}{(2\pi)^3}\psi(k)\mathrm{tr}[\frac{\mathbf{1}}{\sqrt{2}}\Gamma(f)] \qquad (3)$$

and the decay matrix element for an $S = 1$ bound state is

$$\mathcal{M}(B \to f) = \sqrt{\frac{2}{M}} \int \frac{d^3k}{(2\pi)^3}\psi(k)\mathrm{tr}[\frac{\vec{\epsilon}\cdot\vec{\sigma}}{\sqrt{2}}\Gamma(f)]\ . \qquad (4)$$

You can read more details in Peskin and Schroeder, section 5.3. Don't feel obliged to derive these formula; you just need to use them in the below.

(a) Approximating the quark and antiquark in an S-wave nonrelativistic bound state as being at rest, show that the decay width for a spin-1 S-wave bound state to decay to e^+e^- through a virtual photon is

$$\Gamma(B \to e^+e^-) = \frac{16\pi\alpha^2}{M^2}\, Q^2\, |\psi(0)|^2 \;, \tag{5}$$

where

$$\psi(0) = \int \frac{d^3k}{(2\pi)^3}\psi(k) \tag{6}$$

is the value of the coordinate space wavefunction at $r = 0$, and the mass of the electron is set to zero, and Q is the electric charge of the quark. This is identical to Peskin and Schroeder eq. (5.57) except for a color factor of 3. You should explain why this factor is in the right place.

(b) Show explicitly, using the above formulae, that the analogous decay for the spin-0 S-wave bound state, $B \to e^+e^-$, is zero. This is also obvious by angular momentum conservation.

(c) Now consider the decay $B \to gg$, where g is a QCD gluon. Again using the approximation that, in an S-wave bound state, the quark and antiquark are approximately at zero momentum, compute the decay rates for the spin-0 and spin-1 cases. One of these is zero; which, and why?

(d) In real QCD, the quarks have color **3**, but we can also think about bound states of massive fermions with in the **8** of $SU(3)$. Argue (for example, using a Wilson loop) that the QCD potential between a colored particles A and B has the form

$$V(\vec{q}) = \frac{g_s^2}{|\vec{q}|^2} T_A^a T_B^a \tag{7}$$

Evaluate this for (a) a quark in the **3** and an antiquark in the $\overline{\mathbf{3}}$, combining to a QCD singlet, (b) a fermion in the **8** and an antifermion in the **8**, combining to a QCD singlet. (Note that the **8** is a real representation, that is, $\overline{\mathbf{8}}$ is equivalent to **8**.

(e) Group theory says:

$$\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{27} \tag{8}$$

is the decomposition of the product into irreducible representations of $SU(3)$. The two **8**s on the right-hand side are, respectively, symmetric and antisymmetric in the two original **8**s. A member c of the antisymmetric **8** is described by the combination f^{abc}; A member c of the symmetric **8** is described by the combination d^{abc}, where

$$\{t^a, t^b\} = \frac{1}{3}\delta^{ab}\mathbf{1} + d^{abc}t^c \tag{9}$$

Show that, for $\mathbf{8} \times \mathbf{8} \to \mathbf{8}$, the potential is attractive.

(f) For an S-wave bound state of two fermions in the **8** combining to a **1** and in the appropriate spin combination, compute the decay rate for $B \to gg$.

(g) For an S-wave bound state of two fermions in the **8** combining to a symmetric **8** and in the appropriate spin combination, compute the decay rate for $B \to gg$.

(h) For an S-wave bound state of two fermions in the **8** combining to an antisymmetric **8** and in the appropriate spin combination, compute the decay rate for $B \to gg$.

2. In the Standard Model, quarks and leptons cannot obtain a mass without coupling to the Higgs field. However, it is possible to introduce "vectorlike" fermions that can obtain mass on their own. For example, we can introduce heavy leptons

$$E_L = \begin{pmatrix} E_L^+ \\ E_L^0 \end{pmatrix} \qquad E_R = \begin{pmatrix} E_R^+ \\ E_R^0 \end{pmatrix} \tag{10}$$

such that both E_L and E_R have the $SU(2) \times U(1)$ quantum numbers $I = \frac{1}{2}$, $Y = \frac{1}{2}$. Similarly, we can introduce a complex-valued boson field

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}^+ \\ \mathcal{E}^0 \end{pmatrix} \tag{11}$$

with $I = \frac{1}{2}$, $Y = \frac{1}{2}$. In this problem, you will study the properties of these particles, mainly in the limit where their masses are very large.

(a) Write the $SU(2) \times U(1)$ covariant derivatives on the fields E and $\mathcal{E}$ in terms of the Standard Model gauge boson eigenstates $W^\pm$, Z, A. Show that the electric charges of these fields are as given above. Write the mass terms for these fields and show that they are $SU(2) \times U(1)$-invariant. Let the mass for E or $\mathcal{E}$ be M.

(b) Draw the one-loop diagrams that give corrections to the masses of the E^+ and E^0. Write expressions for these diagrams. Show that, if the vector boson propagators have the form, e.g.,

$$\langle Z^\mu(p) Z^\nu(-p) \rangle = \frac{-i}{p^2 - m_Z^2} \Big(g^{\mu\nu} - A p^\mu p^\nu \Big) , \tag{12}$$

then the term involving A gives zero contribution.

(c) Compute the mass shifts of E^+, E^0. You can leave the answer as a 1-parameter integral over the Feynman parameter. Show that the mass *difference* is UV finite.

(d) Take the limit of this mass difference as $m_E \gg m_W$. You should find that it is finite and nonzero. Evalute it numerically (in GeV). Compare this mass difference to the mass of the π^+ meson, $m(\pi^+) = 139.6$ MeV.

(e) Carry out the calculations in parts (b), (c), (d), for the case of $\mathcal{E}^+$, $\mathcal{E}^0$. How does the final answer compare to that in part (d)? Why?

(f) Using the formula

$$\langle 0 | \, j^{\mu a} \, \Big| \pi^b(p) \Big\rangle = i \frac{1}{2} p^\mu f_\pi \delta^{ab} \tag{13}$$

where $a, b = 1, 2, 3$ are $SU(2)$ indices and $j^{\mu a}$ is the $SU(2)$ gauge current, compute the rate of the decay $E^+ \to E^0 \pi^+$. This turns out to be the dominant decay mode of the E^+ when $m_E \gg m_W$, so we can approximate the total decay rate by the rate for this process. Using this estimate, compute the decay length $c\tau$ numerically.