Physics 331 - Final Exam

due March 23, 2018, at noon

This is a take-home final exam. The rules for the exam are: (1) Please do not collaborate with other students in the course or seek other human resources; (2) You may use any references that you find, but please do not actively search the web for the solutions to these problems. Both problems can be solved by just thinking about them using the methods that we have discussed in this course. If you find a very useful reference, though, please cite it in your solution.

The exam has 2 long problems, each worth 50 points. Partial credit will be given. If you make progress, please put it down on paper.

If you find apparent typos on this exam, or if you have any other questions, please notify me immediately at mpeskin@slac.stanford.edu.

Please include with your exam a signed statement that you acknowledge and accept the Stanford Honor Code (as on the cover of a Stanford blue book).

The two problems on this exam deal with "decoupling", the effect on a quantum field theory of a particle whose mass is taken to be very large. There are many interesting and somewhat counterintuitive effects associated with decoupling.

1. This problem deals with decoupling in QED. In QED, we canonically renormalize on the mass shell and define α from the strength of the Coulomb interaction at large distances.

   In this problem, you will need to know that $m_e = 0.51$ MeV and that $m_\mu = 106$ MeV. To solve this problem, consider the ratio of these masses to be sufficiently large that the logarithm of this ratio, $\log(m_\mu/m_e)$, is large and can be treated as an expansion parameter.

   (a) Draw the 1-loop Feynman diagram that contributes to the $a_\mu = (g-2)/2$ of the muon. Show that, at this level, a_μ for the muon equals the corresponding quantity a_e for the electron.

   (b) Draw the 2-loop Feynman diagram by which the electron contributes to a_μ. Show that this diagram is divergent (in a subgraph) but that it is rendered finite when combined with a 1-loop diagram that contains one of the standard QED counterterms.

   (c) Analyze this sum of diagrams. It is easiest to compute and renormalize the divergent subgraph, then put this expression into the final loop integral. Compute the contribution to a_μ proportional to $\alpha^2 \log m_\mu/m_e$. [Hint: Isolate the term containing $\log m_e$ and ignore everything else.]

(d) Now interchange the roles of the muon and the electron. Analyze the leading correction to a_e due to the muon and show that it is no larger than of order $(m_e^2/m_\mu^2)\log m_\mu/m_e$. [The calculation of the coefficient of (m_e^2/m_μ^2) is nontrivial. I will give extra credit to anyone who can obtain it.]

(e) Explain the difference between the behaviors in parts (c) and (d) using intuition from the renormalization group.

(f) Compute the contribution of the electron to the a_μ including all terms of order $\alpha \cdot (\alpha \log(m_\mu/m_e))^n$. To do this, draw the Feynman diagrams that give rise to these terms and sum them up.

(g) If we think of the electron mass as a renormalization scale for α in the computation of the a_μ, then a_μ should satisfy the Callan-Symanzik equation

$$\Big[m_e \frac{\partial}{\partial m_e} + \beta(e)\frac{\partial}{\partial e}\Big] a_\mu(\alpha, m_\mu/m_e) = 0 \ . \tag{1}$$

Explain the logic of this statement. (Looking more carefully, this equation captures all logarithms of m_e but omits terms of order (m_e^2/m_μ^2).) Show that eq. (1) gives an alternative derivation of the result in part (f).

(h) It is known, for the electron $(g-2)$, in QED with electrons and photons only, that

$$a_e = \frac{1}{2}\frac{\alpha}{\pi} + A_2(\frac{\alpha}{\pi})^2 + \cdots \tag{2}$$

where

$$A_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\log 2 + \frac{3}{4}\zeta(3) \approx -0.32848 \ . \tag{3}$$

and

$$\beta(e) = \frac{4}{3}\frac{e^3}{(4\pi)^2} + 4\frac{e^5}{(4\pi)^4} + \cdots \tag{4}$$

Using this data, solve the Callen-Symanzik equation (1) for a_μ to the next order in α and compute the term in a_μ of order $\alpha^3 \log(m_\mu/m_e)$. Draw a few diagrams that contribute to this term.

2. This problem deals with decoupling in QCD. QCD is an $SU(3)$ Yang-Mills theory with 6 flavors of quarks in the **3** representation of $SU(3)$. The quarks have masses that range from $m_u = 2$ MeV for the u quark to $m_t = 173$ GeV for the t quark. It is typically a very good approximation to ignore the masses of the u and d quarks. Then the mass of the proton (a bound state of uud) is almost entirely due to QCD strong interactions.

   QCD is an asymptotically free theory, so we should think of the boundary condition on the theory as occurring at a very large momentum M, where $g(M)$ is small, or, alternatively, where $\alpha_s(M) = g^2(M)/4\pi$ takes a small value. The value of $\alpha_s(q)$ grows for smaller q, and eventually this coupling becomes strong.

(a) Compute the Coulomb potential in QCD between a heavy quark and a heavy antiquark. For example, compute the amplitude for nonrelativistic quark-antiquark scattering and compare to a standard form as on p. 125 of Peskin and Schroeder. Show that, if the quark and antiquark are in a gauge-singlet state

$$\left|Q_i\overline{Q}_j\right\rangle \sim \delta_{ij} \tag{5}$$

where $i, j = 1, 2, 3$ are color indices, then

$$V(\vec{q}) = -g^2 C_2(\mathbf{3})\frac{1}{|\vec{q}|^2} \qquad C_2(\mathbf{3}) = \frac{4}{3}\ . \tag{6}$$

Using the Callan-Symanzik equation for $V(\vec{q})$, show that higher-order corrections will change g in this equation into the solution of the renormalization group (RG) equation for the running coupling constant,

$$\frac{d}{d\log q}g(q) = \beta(g)\ . \tag{7}$$

(b) Solve the RG equation for QCD with n_f massless quarks, using the leading-order β function. You will see that the value of $g(q)$ diverges an a finite value of q. Define the position of this apparent divergence to be Λ. Λ estimates the dynamically generated scale at which the QCD coupling becomes strong. For 5 massless flavors and the boundary condition $\alpha_s(91\ \text{GeV}) = 0.118$, compute Λ (in MeV).

(c) The t quark is very heavy, but it is relevant to the evolution of $g(q)$ for $q > 2m_t$. To understand this effect, compute the QCD Coulomb potential as a function of q including the effect of t quark loops. To do this, modify the running coupling constant obtained in part (b) for 5 massless flavors by including the effect of the t quark vacuum polarization. Include the full m_t-dependence of this diagram.

(d) Notice that, for $q \ll 2m_t$, the running coupling $g(q)$ obtained in part (c) has the form of a solution to the RG equation with 5 massless quarks and that, for $q \gg 2m_t$, the running coupling has the form of a solution to the RG equation with 6 massless quarks. The transition between these two behaviors is smooth. However, if we are not concerned with energies of order m_t, it is convenient to describe the running of $g(q)$ using only RG equations with massless quarks. Here is a way to do that: Consider an artificial $g(q)$ that is exactly a solution to the 5-flavor RG equation for $q < C \cdot m_t$, exactly a solution to the 6-flavor RG equation for $q > C \cdot m_t$, and is continuous (but with a discontinuity in the slope) at $q = C \cdot m_t$. Find C such that this prescription gives the correct values of $g(q)$ for $q \ll 2m_t$ corresponding to the boundary condition that $g(q) = g(M)$ at the renomalization scale $M \gg m_t$.

(e) Using this artificial (but simpler) $g(q)$, compute

$$\frac{\partial \Lambda}{\partial m_t}\ , \tag{8}$$

holding $g(M)$ fixed at the renormalization scale M.

(f) If we ignore the masses of all quarks except t, the proton mass is determined only by the QCD strong interaction scale. Then

$$m_p = A \cdot \Lambda \tag{9}$$

where A is a dimensionless constant that must be computed by some strong-coupling method. In particular, A is independent of m_t. Combining eqs. (8) and (9), compute $\partial m_p / \partial m_t$.

(g) An important effect of the Higgs boson is that it gives rise to the mass of the top quark. In fact,

$$m_t = \frac{y_t}{\sqrt{2}} \langle h \rangle \tag{10}$$

where y_t is a coupling constant and $\langle h \rangle$ is the Higgs field vacuum expectation value after a symmetry-breaking phase transition. If $\langle h \rangle$ changes, the value of the t quark mass changes, and then the value of the proton mass changes. This implies that the Higgs boson couples to the proton via a coupling

$$\mathcal{L} = g_{pph} \, \overline{p} p \; h \ , \tag{11}$$

where

$$g_{pph} = \frac{\partial m_p}{\partial m_t} \cdot \frac{m_t}{\langle h \rangle} \ . \tag{12}$$

From the theory of weak interactions, it is known that $\langle h \rangle = 250$ GeV. Evaluate g_{pph}.

(h) What about the other quarks b and c whose masses are much greater than m_p. Do these give similar, larger, or smaller contributions to the *pph* coupling? What about the quarks u, d, and s whose masses are much smaller than m_p?

(i) From the considerations of parts (g) and (h), estimate

$$\frac{g_{pph}^2}{4\pi} \ . \tag{13}$$

Using this, estimate the scattering cross section at low energy for two fermions of mass about 1 GeV whose coupling to the Higgs boson is similar to that of the proton. This is the size of "direct detection" cross sections seen in "Higgs portal" models of dark matter.