Physics 331 - Final Exam

due March 24, 2017, at 3:00 pm

This is a take-home final exam. The rules for the exam are: (1) Please do not collaborate with other students in the course or seek other human resources; (2) You may use any references that you find, but please do not actively search the web for the solutions to these problems. Your time will be better spent working out the solutions for yourself. If you find a very useful reference, though, please cite it in your solution.

The exam has 1 long problem, worth 100 points in all for parts (a)-(h). Partial credit will be given.

If you find apparent typos on this exam, or if you have any other questions, please notify me immediately at mpeskin@slac.stanford.edu.

1. This problem deals with the renormalization of 4-dimensional quantum field theory models with supersymmetry. You will not need to know what supersymmetry is to do this problem. I hope that this problem will motivate you to learn more about supersymmetric 4-d models — after the exam is over.

   (a) As a first step, I will define the Majorana mass of a fermion. Let

   $$c = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

   Consider the Lagrangian for left-handed chiral fermions:

   $$\mathcal{L} = \psi_i^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2}(m_{ij}\psi_i^T c\psi_j - m_{ij}^* \psi_i^\dagger c\psi_j^*) \ . \tag{2}$$

   Show that (1) by Fermi statistics, m_{ij} must be a symmetric matrix, (2) the Lagrangian is Hermitian, (3), the Lagrangian is Lorentz-invariant (see Peskin and Schroeder, eq. (3.37)); (4) if in the Dirac Lagrangian you substitute

   $$\Psi = \begin{pmatrix} \psi_1 \\ -c\psi_2^* \end{pmatrix} \ , \tag{3}$$

   you recover the Lagrangian (1) in the special case $i, j = 1, 2$ and $m_{ij} = m(\sigma^1)_{ij}$.

   (b) Work out the field equation for the field ψ_i, and look for plane-wave solutions with $p^\mu = (E, 0, 0, p)^\mu$. Show that there is a massive spin $\frac{1}{2}$ fermion with mass equal to each eigenvalue of the matrix m_{ij}.

   In the general case, m_{ij} is called a Majorana mass term. It is easy to calcuate Feynman diagrams with Majorana mass terms and with interactions of the same form. The only subtlety is accounting the minus signs. For this problem set, I recommend that you do that writing the interaction terms that come down from the exponent of the functional integral and making explicit contractions. In the calculations in this problem set, it is correct to perform the fermion spin computations in 4 dimensions while keeping the momenta in d dimensions

(c) Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger i\overline{\sigma}^\mu \partial_\mu \psi + F^* F + (mF\phi - \frac{1}{2} m \psi^T c \psi + h.c.) \tag{4}$$

Show that, integrating over F and F^*, the boson field ϕ and the fermion field ψ obtain the same mass.

(d) Consider the Lagrangian of the *Wess-Zumino model* with coupling constants η_1, η_2,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger i\overline{\sigma}^\mu \partial_\mu \psi + F^* F + (\eta_1 F \phi^2 - \eta_2 \psi^T c \psi \phi + h.c.) \tag{5}$$

To obtain a version of this theory that is easier to work with, integrate out F and F^*. Then compute the 1-loop correction to the boson mass. Show that the quadratic divergence of the boson mass vanishes for $\eta_1 = \eta_2$.

(e) Compute the β functions for η_1 and η_2 in the Wess-Zumino model. Show that the relation $\eta_1 = \eta_2$ is preserved by the renormalization group. It this condition stable or unstable as one evolves toward the infrared?

(f) Supersymmetric Yang-Mills theory is the model

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \lambda^{\dagger a} i\overline{\sigma}^\mu D_\mu \lambda^a \ , \tag{6}$$

where λ^a is a left-handed fermion field in the adjoint representation, and D_μ is the usual gauge-covariant derivative. Using the more general result proven in class, compute the β function for the gauge coupling in this model.

(g) Supersymmetric Yang-Mills theory can be coupled to the Wess-Zumino model by adding to the above Lagrangian

$$\begin{aligned} \mathcal{L} \;&=\; D_\mu \phi^\dagger D^\mu \phi + \psi^\dagger \overline{\sigma}^\mu D_\mu \psi \\ &\quad -\sqrt{2} g_1 (\phi^\dagger \lambda^{Ta} t^a_R c \psi - \psi^\dagger c \lambda^{a*} t^a_R \phi) - \frac{1}{2} g_2^2 (\phi^\dagger t^a_R \phi)^2 \ , \end{aligned} \tag{7}$$

where ϕ and ψ belong to the representation R of the gauge group. Using the more general result proven in class, compute the β function for the gauge coupling in this model.

(h) Compute the β functions for the couplings g_1 and g_2. Show that the condition $g_1 = g_2 = g$ is preserved by the renormalization group.

(i) Extra credit: A property of the computation in (e) is that there is a nice cancellation between a boson loop and a fermion loop. This is more clear in the form where F, F^* are not integrated out. Then there is a nonzero δ_F, but the vertex $\eta_1 F \phi^2$ has $\delta\eta_1 = 0$ for $\eta_1 = \eta_2$. Also recall that $\delta_{\eta_2} = 0$. Consider the more general Lagrangian with multiple (ϕ_i, ψ_i, F_i) fields and

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + \psi_i^\dagger i\overline{\sigma}^\mu \partial_\mu \psi_i + F_i^* F_i + (F_i \frac{dW}{d\phi_i} - \frac{1}{2} \psi_i^T c \psi_j \frac{d^2 W}{d\phi_i d\phi_j} + h.c.) \ , \tag{8}$$

where $W(\phi)$ is a polynomial in the fields ϕ_i. Show that the cancellation generalizes. What is the implication?