Physics 331 - Final Exam
Solutions

a.) The unbroken symmetries are symmetry transformations that leave $\langle\Sigma\rangle$ invariant : $\delta\langle\Sigma\rangle = 0$

The $SU(3)\times SU(2)\times U(1)$ symmetry acts on $\langle\Sigma\rangle$ as:

$$SU(3) \qquad \delta\langle\Sigma\rangle = t^A\langle\Sigma\rangle$$

$$SU(2) \qquad \delta\langle\Sigma\rangle = \langle\Sigma\rangle(-\tau^a)$$

$$U(1) \qquad \delta\langle\Sigma\rangle = -\tfrac{1}{2}\cdot 1\cdot\langle\Sigma\rangle$$

The following combination leave $\langle\Sigma\rangle$ invariant:

$$\delta_a\langle\Sigma\rangle = t^a\langle\Sigma\rangle - \langle\Sigma\rangle\tau^a$$

where $a = 1,2,3$ $\qquad t^a = \left(\begin{array}{c|c} \tau^a & \\ \hline & 0 \end{array}\right)$

$$\delta_1\langle\Sigma\rangle = \sqrt{3}\, t^8\langle\Sigma\rangle - \tfrac{1}{2}\cdot 1\cdot\langle\Sigma\rangle$$

where $t^8 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ so $\operatorname{tr}(t^8)^2 = \tfrac{1}{2}$

This corresponds to an unbroken $SU(2)\times U(1)$ symmetry
If y is the original $U(1)$, the final $U(1)$ is

$$Y = y + \sqrt{3}\, T^8$$

b.) The kinetic energy term for Σ is $\quad D_\mu\Sigma^\dagger D^\mu\Sigma$

If we write $\quad D_\mu\Sigma = \partial_\mu\Sigma - i\,[A\Sigma]$, the mass term for

gge fields is

$$\frac{1}{2} m^2_{AB} A^A_\mu A^{\mu B} = tr\,([A\Sigma])^\dagger [A\Sigma]$$

evaluate with $\Sigma \to \langle\Sigma\rangle = V \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$[A\Sigma] = g_3 A^A_\mu t^A \Sigma - g_2 \Sigma \tau^a B^a_\mu - \frac{1}{2} g_1 C_\mu \Sigma$$

$$tr\,[A\Sigma]^\dagger [A\Sigma]\Big|_{\Sigma\to\langle\Sigma\rangle}$$

$$= tr\,[(g_3 A^A_\mu t^A\langle\Sigma\rangle - g_2\langle\Sigma\rangle\tau^a B^a_\mu - \frac{1}{2}g_1 C_\mu\langle\Sigma\rangle)$$

$$(g_3\langle\Sigma\rangle^\dagger t^B A^{\mu B} - g B^{\mu b}\tau^a\langle\Sigma\rangle^\dagger - \frac{1}{2}g_1 C_\mu\langle\Sigma\rangle^\dagger]$$

$$= tr\,\{ g_3^2 A^A_\mu A^{\mu B}\ \langle\Sigma\rangle\langle\Sigma\rangle^\dagger t^B t^A$$

$$+ g_2^2 B^a_\mu B^{\mu b}\ \langle\Sigma\rangle^\dagger\langle\Sigma\rangle\tau^a\tau^b + g_1^2\ \frac{1}{4}C_\mu C^\mu\ \langle\Sigma\rangle^\dagger\langle\Sigma\rangle$$

$$+ (-g_3 g_2)A^A_\mu B^{\mu a}\ (t^A\langle\Sigma\rangle\tau^a\langle\Sigma\rangle^\dagger + \langle\Sigma\rangle\tau^a\langle\Sigma\rangle^\dagger t^A)$$

equal!

$$+ (-g_3 g_1)A^A_\mu C^\mu\ (t^A\langle\Sigma\rangle\langle\Sigma\rangle^\dagger\cdot\frac{1}{2}\cdot 2)$$

$$+ g_1 g_2 B^a_\mu C^\mu\ \langle\Sigma\rangle\tau^a\langle\Sigma\rangle^\dagger\cdot\frac{1}{2}\cdot 2\ \}$$

Let's first see which of the cross terms are nonzero

$$\mathrm{tr}\ \langle\Sigma\rangle \tau^a \langle\Sigma\rangle^\dagger = \mathrm{tr}\ \langle\Sigma\rangle^\dagger \langle\Sigma\rangle \tau^a = \mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix}\tau^a = 0$$

$$\mathrm{tr}\ t^A \langle\Sigma\rangle\langle\Sigma\rangle^\dagger = \mathrm{tr}\ t^A \left(\begin{array}{cc|c}1&0&0\\0&1&0\\ \hline 0&0&0\end{array}\right)$$

of the 8 SU(3) generators, this is nonzero only for $t^8 = \frac{1}{2\sqrt3}\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}$

$$= \delta^{A,8}\frac{1}{\sqrt3}$$

$$\mathrm{tr}\left(t^A \langle\Sigma\rangle \tau^a \langle\Sigma\rangle^\dagger\right) = \mathrm{tr}\ t^A \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\tau^a\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)$$

if $\quad t^A = \left(\begin{array}{c|c}\tau^b&0\\ \hline 0\ 0&0\end{array}\right)$ this is $\mathrm{tr}\ \tau^b\cdot 1\cdot\tau^a\cdot 1 = \delta^{ab}\cdot\frac12$

$t^A = \left(\begin{array}{c|c}&\xi\\ \hline \xi^\dagger&\end{array}\right) \qquad \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\left(\begin{array}{c|c}&\xi\\ \hline \xi^\dagger&\end{array}\right)\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} = 0$

$t^A = t^8 \qquad = \mathrm{tr}\ \frac{1}{2\sqrt3}\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\tau^a\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$

$$= \frac{1}{2\sqrt3}\,\mathrm{tr}\ 1\cdot 1\cdot \tau^a\cdot 1 = 0$$

So $\mathcal{A}^A$ mixes with C for $A = 8$

mixes with B^a for $A = 1,2,3$.

Now for the diagonal terms:

$$\mathrm{tr}\ \langle\Sigma\rangle^\dagger\langle\Sigma\rangle = \mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix} = 2$$

$$\mathrm{tr}\ \langle\Sigma\rangle^\dagger\langle\Sigma\rangle\tau^a\tau^b = \mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix}\tau^a\tau^b = \frac12\delta^{ab}$$

$$\mathrm{tr}\ \langle\Sigma\rangle\langle\Sigma^\dagger\rangle t^A t^B = \mathrm{tr}\left(\begin{array}{cc|c}1&0&0\\0&1&0\\ \hline 0&0&0\end{array}\right)t^A t^B$$

for $A,B = a,b = 1,2,3$

$$= \mathrm{tr}\left(\begin{array}{c|c} \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \\ \hline & \end{array}\right)\left(\begin{array}{c|c} \tau^a & \\ \hline & \end{array}\right)\left(\begin{array}{c|c} \tau^b & \\ \hline & \end{array}\right) = \tfrac{1}{2}\delta^{ab}$$

$A = B = 8$

$$= \mathrm{tr}\left(\begin{array}{c|c} \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \\ \hline & \end{array}\right)\frac{1}{12}\begin{pmatrix}1&&\\&1&\\&&4\end{pmatrix} = \frac{1}{6}$$

$$t^A = \frac{1}{2}\left(\begin{array}{c|c} & \begin{smallmatrix}1\\0\end{smallmatrix} \\ \hline \begin{smallmatrix}1&0\end{smallmatrix} & \end{array}\right) = t^B \qquad A = B = 4$$

$$= \mathrm{tr}\left(\begin{array}{c|c} \begin{smallmatrix}1&0\\0&\end{smallmatrix} & \\ \hline & \end{array}\right)\frac{1}{4}\left(\begin{array}{c|c} \begin{smallmatrix}1&\\&0\end{smallmatrix} & \\ \hline & 1\end{array}\right) = \frac{1}{4}$$

$t^5 = \frac{1}{2}\left(\begin{array}{c|c} & -i \\ \hline i & \end{array}\right)$ $\quad t^6 = \frac{1}{2}\left(\begin{array}{c|c} & \begin{smallmatrix}0\\1\end{smallmatrix} \\ \hline \begin{smallmatrix}0&1\end{smallmatrix} & \end{array}\right)$ $\quad t^7 = \frac{1}{2}\left(\begin{array}{c|c} & \begin{smallmatrix}\\-i\end{smallmatrix} \\ \hline \begin{smallmatrix}&i\end{smallmatrix} & \end{array}\right)$ give the

same answer.

So

$$\frac{1}{2} m_{AB} A^A A^B = g_3^2 \Big\{ \frac{1}{2} \underbrace{A^a_\mu A^{\mu a}}_{a=1,2,3} + \frac{1}{6} A^8_\mu A^{\mu 8} + \frac{1}{4} \underbrace{A^i_\mu A^{\mu i}}_{i=4567} \Big\} V^2$$

$$+ g_2^2 \Big\{ \frac{1}{2} B^a_\mu B^{\mu b} \Big\} \cdot V^2$$

$$+ g_1^2 \Big\{ \frac{1}{2} C_\mu C^\mu \Big\} \cdot V^2$$

$$- 2 g_2 g_3 \left(\frac{1}{2} A^a_\mu B^{\mu a} \right) V^2$$

$$- 2 g_1 g_3 \left(\frac{1}{2\sqrt{3}} A^8_\mu C^\mu \right) V^2$$

$$= \frac{1}{2}\left(g_1 C^\mu - \frac{g_3}{\sqrt{3}} A^8_\mu \right)^2 V^2 + \frac{1}{2}\left(g_2 B^a_\mu - g_3 A^a_\mu \right)^2$$

$$+ \frac{1}{2} \cdot \frac{g_3^2}{2} (A^i_\mu)^2 V^2$$

so the massive vector bosons are:

$$\mathcal{Z}_\mu = \frac{g_1 C^\mu - g_3 A_\mu^8/\sqrt{3}}{[g_1^2 + g_3^2/3]^{1/2}}$$

$$\mathcal{W}_\mu^a = \frac{g_2 B_\mu^a - g_3 A_\mu^a}{[g_2^2 + g_3^2]^{1/2}}$$

$$X_\mu^i = A_\mu^i$$

with masses

$$m^2(\mathcal{Z}) = (g_1^2 + g_3^2/3)\, V^2$$

$$m^2(\mathcal{W}) = (g_2^2 + g_3^2)\, V^2$$

$$m^2(X) = \frac{g_3^2}{2} V^2$$

the orthogonal vector boson eigenstates are massless:

$$B_\mu = \frac{g_3/\sqrt{3}\, C^\mu + g_1 A_\mu^8}{[g_1^2 + g_3^2/3]^{1/2}}$$

$$A_\mu^a = \frac{g_3 B^{\mu a} + g_2 A^{\mu a}}{[g_2^2 + g_3^2]^{1/2}}$$

It is clear that $\mathcal{W}_\mu^a$ A_μ^a are SU(2) triplets,

$\mathcal{Z}_\mu, B_\mu$ are SU(2) singlets, X_μ^i are two SU(2) 2's.

B_μ^a and C_μ have $y = 0$ $T^8 = 0$.

A_μ^A have $y=0$, and T^8 acts on these as $[t^8, t^A]$

Then $Y = y + \sqrt{3}\,T^8 = 0 + \frac{1}{2}\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix} = \begin{pmatrix}\frac12 & & \\ & \frac12 & \\ & & -1\end{pmatrix}$

this commutes with A_μ^a A_μ^8 $\rightarrow$ $Y = 0$

the X_μ^i form pairs with $Y = \pm 3/2$

so under the unbroken $SU(2)\times U(1)$:

B_μ	$(1, 0)$	A_μ^a	$(3, 0)$
Z_μ	$(1, 0)$	W_μ^a	$(3, 0)$

X_μ^i $(2, 3/2) + (2, -3/2)$

c.) The covariant derivative on quark and lepton fields is

$$D_\mu = \partial_\mu - i g_2 B_\mu^a \tau^a - i g_1 C_\mu Y$$

$$= \partial_\mu - i g_2 \left(\frac{g_3 A_\mu^a + g_2 W_\mu^a}{[g_3^2 + g_2^2]^{1/2}}\right)\tau^a - i g_1\left(\frac{\frac{g_3}{\sqrt{3}} B_\mu + g_1 Z_\mu}{[g_3^2/3 + g_1^2]^{1/2}}\right) Y$$

$$= \partial_\mu - i\left[\frac{g_2 g_3}{[g_2^2 + g_3^2]^{1/2}}\right] A_\mu^a \tau^a - i\left[\frac{g_1 g_3/\sqrt{3}}{[g_1^2 + g_3^2/3]^{1/2}}\right] B_\mu Y$$

$$- i\left(\frac{g_2^2}{[g_2^2 + g_3^2]^{1/2}}\right) W_\mu^a \tau^a - i\left[\frac{g_1^2}{[g_3^2/3 + g_1^2]^{1/2}}\right] Z_\mu Y$$

The second line of this expression contains heavy bosons. The first line contains the $SU(2)\times U(1)$ bosons that survive to lower energy. The $SU(2)\times U(1)$ couplings are:

$$g = \frac{g_2 g_3}{[g_2^2 + g_3^2]^{1/2}} \qquad g' = \frac{g_1 g_3/\sqrt{3}}{[g_1^2 + g_3^2/3]^{1/2}}$$

If $\quad g_1, g_2 \gg g_3$

$$g \approx g_3 \qquad g' \approx g_3/\sqrt{3}$$

$$\sin^2\theta_W = \frac{g'^2}{g^2+g'^2} \cong \frac{1}{4}$$

d.) Evolve this result to energies Q below the scale of spontaneous breaking the $SU(2)$ and $U(1)$ using

$$\alpha^{-1}(Q) = \alpha^{-1}(M) + \frac{b}{2\pi}\log\frac{Q}{M}$$

$$b = \frac{4}{3}C_2(G) - \frac{2}{3}\sum_{\underline{\text{chiral}}\text{ fermion}} C(r) - \frac{1}{3}\sum_{\substack{\text{complex}\\ \text{bosons}}} C(r)$$

for $SU(2)$:

$$C_2(G) = 2 \qquad C(r) = \tfrac{1}{2} \text{ for one doublet;}$$

we need to sum over $\quad 3\times\left\{\binom{u}{d}\binom{c}{s}\binom{t}{b}\right\} + 1\cdot\left\{\binom{\nu}{e}\binom{\nu}{\mu}\binom{\nu}{\tau}\right\} +$ Higgs.

$$b(SU(2)) \equiv b_2 = \frac{22}{3} - \frac{2}{3}\cdot\frac{1}{2}\cdot(3+1)\cdot 3 - \frac{1}{3}\cdot\frac{1}{2}$$

$$= \frac{22}{3} - 4 - \frac{1}{6} = \frac{19}{6}$$

$$b(U(1))_Y \equiv b_1 = -\frac{2}{3}\cdot\{\mathrm{tr}\,Y^2\} - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2\cdot 2$$

$$\mathrm{tr}\,Y^2 = \left[3\cdot 2\cdot\left(\frac{1}{6}\right)^2 + 3\left(-\frac{1}{3}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 2\cdot\left(-\frac{1}{2}\right)^2 + 1^2\right]\cdot 3 \leftarrow \text{3 generations}$$

$$= \frac{10}{3}\cdot 3 = 10$$

$$b_1 = -\frac{20}{3} - \frac{1}{6} = -\frac{41}{6}$$

So

$$\alpha_w = \frac{g^2}{4\pi} \qquad \alpha_w' = \frac{g'^2}{4\pi}$$

$$\alpha_w^{-1}(m_Z) = \alpha_w^{-1}(M) + \frac{19}{12\pi}\lg\frac{m_Z}{M}$$

$$\alpha_w'^{-1}(m_Z) = \alpha_w'^{-1}(M) - \frac{41}{12\pi}\lg\frac{m_Z}{M}$$

Now, at M

$$\alpha_w \simeq \frac{g_3^2}{4\pi} = \alpha_3 \qquad \alpha_w' \simeq \frac{g_3^2}{12\pi} \equiv \alpha_3/3$$

So $$\alpha_W^{-1}(m_Z) = \alpha_3^{-1}(M) - \frac{19}{12\pi}\lg\frac{M}{m_Z}$$

$$\alpha_W'^{-1}(m_Z) = 3\,\alpha_3^{-1}(M) + \frac{41}{12\pi}\lg\frac{M}{m_Z}$$

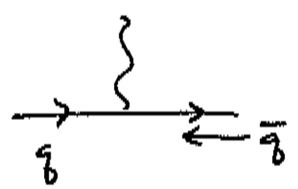
$$-3\,\alpha_W^{-1}(m_Z) + \alpha_W'^{-1}(m_Z) = +\frac{98}{12\pi}\lg\frac{M}{m_Z}$$

using $\alpha_W(m_Z) = \frac{1}{29.6}$ $\quad \alpha_W'(m_Z) = \frac{1}{98.5}$

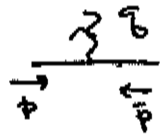
$$\lg\frac{M}{m_Z} = 3.7 \qquad M = 42.\cdot m_Z \cong 3.8 \text{ TeV}$$

$$\alpha_3 = \frac{g_3^2}{4\pi} = \frac{1}{31.4} \qquad (g_3 \cong 0.63)$$

e.) The heavy vector bosons W_μ^a and Z_μ appear in the covariant derivative on quark fields. So they couple to $q\bar{q}$ and can be produced in the Drell-Yan process.



If the vertex is $\quad \Gamma^\mu = i\,G\,\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)$, we can compute the cross-section:



$$= \quad i\mathcal{M} = i\, G \quad \bar{v}(\bar{p})\, \gamma^\mu \left(\tfrac{1-\gamma^5}{2}\right) u(p) \cdot \varepsilon_\mu^*(q)$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{4} G^2 \; \mathrm{tr}\left[\bar{\not p}\, \gamma^\mu \left(\tfrac{1-\gamma^5}{2}\right) \not p\, \gamma^\nu \left(\tfrac{1-\gamma^5}{2}\right)\right] \sum \varepsilon_\mu^*(q)\, \varepsilon_\nu(q)$$

$$= \frac{1}{4} G^2\, \mathrm{tr}\left[\bar{\not p}\, \gamma^\mu \not p\, \gamma^\nu \left(\tfrac{1-\gamma^5}{2}\right)\right] \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_V^2}\right]$$

$$= \frac{1}{4} G^2 \cdot 2 \left[\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu - g^{\mu\nu}\, p\cdot\bar{p}\right] \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_V^2}\right]$$

$$= \frac{1}{4} G^2 \cdot 2 \cdot (4-2)\; p\cdot\bar{p}$$

[note that $\quad q_\mu \left[\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu - g^{\mu\nu}\, p\cdot\bar{p}\right] = \left[q\cdot\bar{p}\; p^\nu + q\cdot p\; \bar{p}^\nu - q^\nu\, p\cdot\bar{p}\right]$

$q = p + \bar{p}$ $\qquad = \left[p\cdot\bar{p}\; p^\nu + \bar{p}\cdot p\; \bar{p}^\nu - (p+\bar{p})^\nu\, p\cdot\bar{p}\right] = 0$.]

$q^2 = m_V^2 = 2p\cdot\bar{p}$

$$= G^2\, p\cdot\bar{p} \quad = \quad \frac{1}{2} G^2 m_V^2$$

$$\sigma(q+\bar{q} \to V) = \frac{1}{2s} \cdot \int \frac{d^3 q}{(2\pi)^4} \frac{1}{2E_q} (2\pi)^4 \delta(p+\bar{p}-q) \left(\frac{1}{4}\sum |\mathcal{M}|^2\right)$$

$$\cdot \left[\frac{1}{3}\cdot\frac{1}{3}\cdot \sum_{\text{colors}} 1\right] \leftarrow \text{color factor of } \tfrac{1}{3}$$

$$= \frac{1}{2m_V^2} \cdot \int \frac{d^4 q}{(2\pi)^4}\, 2\pi\, \delta(q^2 - m_V^2)\, (2\pi)^4 \delta(p+\bar{p}-q) \cdot \frac{1}{2} G^2 m_V^2 \cdot \frac{1}{3}$$

$$= \frac{\pi}{6} G^2\, \delta(s - m_V^2)$$

again:

$$\sigma(q+\bar{q}\to V) = \frac{\pi}{6}\, G^2\, \delta(s-m_V^2) = \frac{2\pi^2}{3}\,\frac{G^2}{4\pi}\,\delta(s-m_V^2)$$

<u>for W_μ^3</u> $\quad u_L \bar{u}_R \to W_\mu^3$ has $G = \frac{1}{2} g_W$

$\quad d_L \bar{d}_R \to W_\mu^3$ has $G = -\frac{1}{2} g_W$

for W_μ^+ $\quad u_L \bar{d}_R \to W_\mu^+$ has $G = \frac{g_W}{\sqrt{2}}$

for W_μ^- $\quad d_L \bar{u}_R \to W_\mu^-$ has $G = \frac{g_W}{2}$

where, from p. 6, $\quad g_W = \frac{g_2^2}{[g_2^2 + g_3^2]^{1/2}} \approx g_2$

<u>for Z_μ</u>

$u_L \bar{u}_L \to Z_\mu$ has $G = \frac{1}{6} g_Z$

$u_R \bar{u}_R \to Z_\mu$ has $G = \frac{2}{3} g_Z$

these are independent processes which <u>add</u> in σ

$d_L \bar{d}_L \to Z$ has $G = \frac{1}{6} g_Z$

$d_R \bar{d}_R \to Z$ has $G = -\frac{1}{3} g_Z$

where, from p. 6 $\quad g_Z = \frac{g_1^2}{[g_1^2 + g_3^2/3]^{1/2}} \approx g_1$

so! If $\alpha_W = \frac{g_2^2}{4\pi} \frac{g_2^2}{[g_2^2 + g_3^2]}$ $\alpha_Z = \frac{g_1^2}{4\pi} \frac{g_1^2}{[g_1^2 + g_3^2/3]}$

$\sigma(pp \to W^3 + X)$

$$= \int dx_1 dx_2 \Big\{ f_u(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_u(x_2) + f_d(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_d(x_2) \Big\}$$

$$\cdot \frac{\pi^2}{6} \alpha_W \, \delta(x_1 x_2 s - m_W^2)$$

$\sigma(pp \to W^+ + X)$

$$= \int dx_1 dx_2 \left[f_u(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_u(x_2) \right] \cdot \frac{\pi^2}{3} \alpha_W \, \delta(x_1 x_2 s - m_W^2)$$

$\sigma(pp \to W^- + X)$

$$= \int dx_1 dx_2 \left[f_d(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_d(x_2) \right] \frac{\pi^2}{3} \alpha_W \, \delta(x_1 x_2 s - m_W^2)$$

$\sigma(pp \to Z + X)$

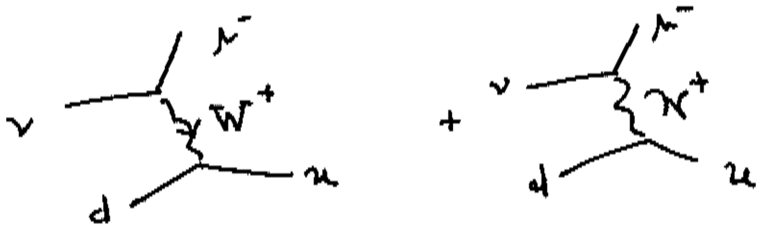
$$= \int dx_1 dx_2 \Big\{ \left[f_u(x_1) f_{\bar{u}}(x_2) + f_{\bar{u}}(x_1) f_u(x_2) \right] \cdot \frac{17}{36} + \left[f_d(x_1) f_{\bar{d}}(x_2) + f_{\bar{d}}(x_1) f_d(x_2) \right] \cdot \frac{5}{36} \Big\}$$

$$\cdot \frac{2\pi^2 \alpha_Z}{3} \, \delta(x_1 x_2 s - m_Z^2)$$

f.) Looking back again at p. 6, we have for the $\mathcal{W}^+$ and W^+ coupling to u, d: $\quad W^+ = \frac{A^1 - iA^2}{\sqrt{2}}$ is the normal W.

$$\delta\mathcal{L} = \frac{g}{\sqrt{2}} W_\mu^+ \, \bar{u}_L \gamma^\mu d_L + \frac{g_{\mathcal{W}}}{\sqrt{2}} \mathcal{W}_\mu^+ \, \bar{u}_L \gamma^\mu d_L$$

and similarly for (ν_μ, μ). Adding $\qquad g_{\mathcal{W}} = \frac{g_2^2}{[g_2^2 + g_3^2]^{1/2}}$ from p. 11



we find the usual expression for ν deep inelastic scattering with

$$\frac{g^2}{2m_W^2} \longrightarrow \frac{g^2}{2m_W^2} + \frac{g_{\mathcal{W}}^2}{2m_{\mathcal{W}}^2}$$

$$= \frac{g^2}{2m_W^2}\left[1 + \frac{g_{\mathcal{W}}^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}\right]$$

so

$$\frac{d^2\sigma}{dx\,dy}(\nu p \to \mu^- + X) = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x(1-y)^2 f_{\bar{u}}(x)\right]$$

$$\cdot \left[1 + \frac{g_{\mathcal{W}}^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}\right]^2$$

the factor in the 2nd line is

$$\approx 1 + 2\frac{g_{\mathcal{W}}^2}{g^2}\frac{m_W^2}{m_{\mathcal{W}}^2}$$

where the correction term is

$$2\,\frac{g_{\mathcal{W}}^2}{g^2}\,\frac{m_W^2}{m_{\mathcal{W}}^2} \cong 2\,\frac{g_2^2}{g^2}\cdot\frac{m_W^2}{m_{\mathcal{W}}^2}$$

$$\sim 2\cdot\frac{g_2^2/4\pi}{\alpha_w}\cdot\left(\frac{80\text{ GeV}}{4000\text{ GeV}}\right)^2$$

If $g_2^2/4\pi \sim \mathcal{O}(1)$

$$\sim 2\%\cdot\left(\frac{g_2^2}{4\pi}\right)$$

so if g_2 is very large, we might be sensitive to this effect if we can measure $\sigma(\nu p\to\mu^- X)$ to the $\underline{1\%}$ level. [Unfortunately, uncertainties from the strong interactions are also at this level.]

g.) The decay of the $\mathcal{W}^+$ to $u\bar{d}$ and other quark or lepton pairs is easy to work out from the coupling on p. 13

$$= i\,\frac{g_{\mathcal{W}}}{\sqrt{2}}\,\bar{u}_L(p)\,\gamma_\mu\, v_L(\bar{p})\,\epsilon_\mu(q)$$

In fact, this is just a cross of the matrix element computed on p. 10

$$\sum_{\text{spins}}|M|^2 = 2G^2m_W^2\cdot 2\left(\frac{g_{\mathcal{W}}}{\sqrt{2}}\right)^2 m_{\mathcal{W}}^2$$

The spin average is now over the spin states of the W, and the color factor is $\underline{3}$, so

$$\frac{1}{3}\sum_{\substack{spin,\\ color}} |M|^2 = \underbrace{3}_{\text{for quarks}} \cdot \frac{1}{3} \cdot g_W^2 m_W^2$$

then

$$\Gamma(W^+ \to \ell^+ \nu) = \frac{1}{2m_W} \frac{1}{8\pi} \cdot \frac{1}{3} g_W^2 m_W^2$$

$$= \frac{\alpha_W}{12} m_W$$

$$\Gamma(W^+ \to u\bar{d}) = 3 \cdot \frac{\alpha_W}{12} m_W \qquad \underline{\alpha_W \text{ on p.12}}$$

and this should be summed over 3 generations (including $W \to t\bar{b}$)

To work out $W^+ \to V_1 V_2$, we need the 3-vector-boson vertex. The terms in W^+ come from both the SU(2) and SU(3) 3-vector terms.

The SU(2) terms come from

$$-\frac{1}{4}(F_{\mu\nu}^a)^2 \to -g_2 \epsilon^{abc} \partial_\mu B_\nu^a B^{\mu b} B^{\nu c}$$

$$= -g_2 \{ \partial_\mu B_\nu^1 (B^{\mu 2} B^{\nu 3} - B^{\nu 2} B^{\mu 3}) + \partial_\mu B_\nu^2 (B^{\mu 3} B^{\nu 1} - B^{\nu 3} B^{\mu 1})$$
$$+ \partial_\mu B_\nu^3 (B^{\mu 1} B^{\nu 2} - B^{\nu 1} B^{\mu 2}) \}$$

now write $\quad B^+_\mu = \frac{1}{\sqrt{2}}(B^1_\mu - i B^2_\mu) \qquad B^-_\mu = \frac{1}{\sqrt{2}}(B^1_\mu + i B^2_\mu)$

or $\quad B^1_\mu = \frac{1}{\sqrt{2}}(B^+_\mu + B^-_\mu) \qquad B^2_\mu = \frac{i}{\sqrt{2}}(B^+_\mu - B^-_\mu)$

$$\begin{aligned} = -g_2 \frac{i}{2} \Big\{ & \partial_\mu (B^+_\nu + B^-_\nu)\left[(B^{\mu+} - B^{\mu-}) B^{\nu 3} - (B^{\nu+} - B^{\nu-}) B^{\mu 3}\right] \\ & + \partial_\mu (B^+_\nu - B^-_\nu)\left[B^{\mu 3}(B^{\nu+} + B^{\nu-}) - B^{\nu 3}(B^{\mu+} + B^{\mu-})\right] \\ & + \partial_\mu B^3_\nu \left[(B^{\mu+} + B^{\mu-})(B^{\nu+} - B^{\nu-}) - (B^{\mu+} - B^{\mu-})(B^{\nu+} + B^{\nu-})\right]\Big\} \end{aligned}$$

$$\begin{aligned} = -\frac{ig_2}{2} \Big\{ & \partial_\mu B^+_\nu \left[-2 B^{\mu-} B^{\nu 3} + 2 B^{\nu-} B^{\mu 3}\right] \\ & + \partial_\mu B^-_\nu \left[2 B^{\mu+} B^{\nu 3} - 2 B^{\nu+} B^{\mu 3}\right] \\ & + \partial_\mu B^3_\nu \left[2 B^{\mu-} B^{\nu+} - 2 B^{\mu+} B^{\nu-}\right] \Big\} \end{aligned}$$

$$\begin{aligned} = -ig_2 \Big\{ & \partial_\mu B^+_\nu \left[B^{\mu 3} B^{\nu-} - B^{\mu-} B^{\nu 3}\right] \\ & + \partial_\mu B^-_\nu \left[B^{\mu+} B^{\nu 3} - B^{\mu 3} B^{\nu+}\right] \\ & + \partial_\mu B^3_\nu \left[B^{\mu-} B^{\nu+} - B^{\mu+} B^{\nu-}\right] \Big\} \end{aligned}$$

we need a W^+ field to annihilate the W^+, so let

$$B^{\mu+} \to \frac{g_2}{[g_2^2 + g_3^2]^{1/2}} W^{\mu+}$$

then

$$B^{\mu-} \to \frac{g_3}{[g_2^2 + g_3^2]^{1/2}} W^{\mu-} \qquad B^{\mu 3} \to \frac{g_3 A^3 + g_2 W^3}{[g_2^2 + g_3^2]^{1/2}}$$

(creates W^+)

It is possible that $m(\chi^3) < m(\chi^+)$ but in any event these states are almost degenerate so the the phase space for

$$\chi^+ \to \chi^3 + W^+$$

will be very small. I will ignore this decay from here on.

on the other hand, $A^3 = c_\omega Z + s_\omega A$ $\qquad s_\omega = \sin\Theta_\omega = \frac{g'}{\sqrt{g^2+g'^2}}$

so!

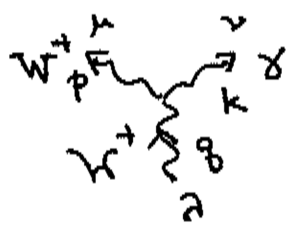
$$= -i\, g_2 \frac{g_2}{[g_2^2+g_3^2]^{1/2}} \cdot \frac{g_3}{[g_2^2+g_3^2]^{1/2}} \cdot \frac{g_3}{[g_2^2+g_3^2]^{1/2}} \cdot s_\omega \cdot$$

$$\cdot \{ \partial_\mu W_\nu^+ [A^\mu W^{\nu -} - W^{\mu -} A^\nu] + \partial_\mu W_\nu^- [\chi^{\mu +} A^\nu - A^\mu \chi^{\nu +}]$$

$$+ \partial_\mu A_\nu [W^{\mu -} \chi^{\nu +} - \chi^{\mu +} W^{\nu -}] \}$$

$$+ (s_\omega A \to c_\omega Z)$$

this gives the vertex:



$$= + \frac{g_2^2 g_3^2}{[g_2^2+g_3^2]^{3/2}} s_\omega$$

$$\cdot \{ -iq^\nu g^{\mu\lambda} + iq^\mu g^{\nu\lambda} + ip^\lambda g^{\mu\nu} - i p^\nu g^{\mu\lambda}$$

$$+ik^\mu g^{\nu\lambda} - i k^\lambda g^{\mu\nu} \}$$

$$= +i \frac{g_2^2 g_3^2}{[g_2^2+g_3^2]^{3/2}} s_\omega \left[g^{\mu\nu}(p-k)^\lambda + g^{\nu\lambda}(q+k)^\mu + g^{\lambda\mu}(-q-p)^\nu \right]$$

and similarly for Z with c_W. There is also an SU(3) contribution to this vertex

$$-\tfrac{1}{4}(F^A_{\mu\nu})^2 \rightarrow -g_3 f^{ABC}\,\partial_\mu A^A_\nu A^{\mu B} A^{\nu C}$$

For $A,B,C = 1,2,3$ $\quad (t^A = \begin{pmatrix} \tau & \\ & 0 \end{pmatrix})$ $\quad f^{ABC} = \epsilon^{ABC}$ and we get

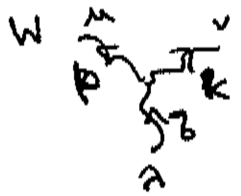
$$= \quad i g_3 \{ \partial_\mu A^+_\nu [A^{\mu 3} A^{\nu -} - \cdots]$$

now

$$A^{\mu +} \rightarrow \frac{-g_3}{[g_2^2+g_3^2]^{1/2}} W^{\mu +}$$

$$A^{\mu -} \rightarrow \frac{g_2}{[g_2^2+g_3^2]^{1/2}} W^{\mu -} \qquad A^{\mu 3} \rightarrow \frac{g_2}{[g_2^2+g_3^2]^{1/2}} A^3 = \frac{g_2}{[g_2^2+g_3^2]^{1/2}}(c_W Z + s_W A)$$

so the SU(3) vertex is



$$= \quad + g_3 \left[\frac{-g_3}{[g_2^2+g_3^2]^{1/2}}\right]\left[\frac{g_2}{[g_2^2+g_3^2]^{1/2}}\right]\left[\frac{g_2}{[g_2^2+g_3^2]^{1/2}}\right] \cdot s_W \quad \cdot \{\ \}$$

$$= \quad +i\,\frac{(-g_2^2 g_3^2)}{[g_2^2+g_3^2]^{1/2}}\, s_W \left[g^{\mu\nu}(p-k)^\lambda + \cdots \right]$$

and exactly cancels the result on p. 17. In all

$=$

$= \quad 0 \quad$ at tree level.

Why is this? Since the vertex is $\propto f^{abc}$, the $W^+ A^3$ vertex can only involve generators in the same subgroup — and this is the <u>unbroken</u> SU(2) subgroup of the gauge group.

However, the SU(3) vertex also allows

$$W^+ \to X^{+2}\,\bar{X}^{-}$$

Let's work this out. The matrices

$$\left(\begin{array}{cc|c} & & 1 \\ & & 0 \\ \hline 0 & 0 & \end{array}\right) \quad \text{have} \quad T^3 = +\tfrac{1}{2} \quad Y = \tfrac{3}{2} \quad \Rightarrow \quad Q = 2$$

$$\left(\begin{array}{cc|c} & & 0 \\ & & 1 \\ \hline & & \end{array}\right) \quad T^3 = -\tfrac{1}{2} \quad Y = 3/2 \quad \Rightarrow \quad Q = 1$$

$$\left(\begin{array}{cc|c} & & \\ & & \\ \hline 1 & 0 & \end{array}\right) \quad T^3 = -\tfrac{1}{2} \quad Y = -3/2 \quad \Rightarrow \quad Q = -2$$

$$\left(\begin{array}{cc|c} & & \\ & & \\ \hline 0 & 1 & \end{array}\right) \quad T^3 = +1/2 \quad Y = -3/2 \quad \Rightarrow \quad Q = -1$$

so write the covariant derivative in SU(3) as $\hat{W}^+_\mu = A^{3+}_\mu = \frac{-g_3}{(\ \ )^{1/2}} W^+$

$$D_\mu = \partial_\mu - i\frac{g_3}{\sqrt{2}}\Bigg[\hat{W}^+_\mu \left(\begin{array}{c|c} \sigma^+ & \\ \hline & \end{array}\right) + X^{+2}_\mu \left(\begin{array}{cc|c} & & 1 \\ & & 0 \\ \hline & & \end{array}\right) + X^{-2}_\mu \left(\begin{array}{cc|c} & & \\ \hline 1 & 0 & \end{array}\right)$$

$$+\, X^+_\mu \left(\begin{array}{cc|c} & & 0 \\ & & 1 \\ \hline & & \end{array}\right) + X^-_\mu \left(\begin{array}{cc|c} & & \\ \hline 0 & 1 & \end{array}\right)\Bigg]$$

$$-ig_3 F^A_{\mu\nu} t^A = [D_\mu, D_\nu]$$

$$= -ig_3\left[(\partial_\mu A^A_\nu - \partial_\nu A^A_\mu)t^A - ig_3\,[A^A_\mu t^A, A^B_\nu t^B]\right]$$

$$= -ig_3 \left\{ \tfrac{1}{\sqrt{2}}(\partial_\mu \hat{W}_\nu^+ - \partial_\nu \hat{W}_\mu^+) \left(\begin{array}{c|c} \sigma^+ & \\ \hline & \end{array}\right) + \tfrac{1}{\sqrt{2}}(\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2}) \left(\begin{array}{c|c} & \\ \hline 10 & \end{array}\right) + \cdots \right.$$

$$-\frac{ig_3}{2} \Big[(\hat{W}_\mu^+ X_\nu^+ - \hat{W}_\nu^+ X_\mu^+) \left(\begin{array}{c|c} & \begin{smallmatrix}1\\0\end{smallmatrix} \\ \hline & \end{array}\right)$$

$$+ (\hat{W}_\mu^+ X_\nu^{-2} - \hat{W}_\nu^+ X_\mu^{-2}) \left(\begin{array}{c|c} & \\ \hline 0\,{-1} & \end{array}\right)$$

$$+ (X_\mu^{-2} X_\nu^+ - X_\nu^{-2} X_\mu^+) \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \cdots$$

$$-\tfrac{1}{4}(F_{\mu\nu}^A)^2 = -\tfrac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}^A t^A)^2$$

$$= -\mathrm{tr}\,[(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A) t^A]\,[A^{\mu B} t^B, A^{\nu C} t^C]$$

$$= -\tfrac{1}{\sqrt{2}} \cdot \left(\tfrac{-ig_3}{2}\right) \cdot \Big\{ (\partial_\mu \hat{W}_\nu^+ - \partial_\nu W_\mu^+)(X_\mu^{-2} X_\nu^+ - X_\nu^{-2} X_\mu^+) \cdot (-1)$$

$$+ (\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2})(\hat{W}_\mu^+ X_\nu^+ - X_\mu^+ \hat{W}_\nu^+) \cdot 1$$

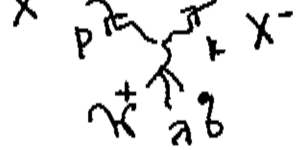
$$+ (\partial_\mu X_\nu^+ - \partial_\nu X_\mu^+)(\hat{W}_\mu^+ X_\nu^{-2} - \hat{W}_\nu^+ X_\mu^{-2}) \cdot (-1) \Big\}$$

$$= -i\frac{g_3}{2\sqrt{2}} \Big\{ (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(X^{\mu -2} X^{\nu +} - X^{\mu +} X^{\nu -2})$$

$$+ (\partial_\mu X_\nu^{-2} - \partial_\nu X_\mu^{-2})(X^{\mu +} W^{\nu +} - W^{\mu +} X^{\nu +})$$

$$+ (\partial_\mu X_\nu^+ - \partial_\nu X_\mu^+)(W^{\mu +} X^{\nu -2} - X^{\mu -2} W^{\nu +}) \Big\}$$

comparing to p. 17, this gives:



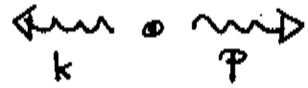
$$= \frac{-ig_3}{2\sqrt{2}} \left(\frac{-g_3}{[g_2^2 + g_3^2]^{1/2}}\right) \left[g^{\mu\nu}(p-k)^\lambda + g^{\nu\lambda}(q+k)^\mu + g^{\mu\lambda}(q-p)^\nu \right]$$

Now evaluate this as a matrix element between on-shell vector boson states:

$$i\mathcal{M} = \frac{i\, g_3^2}{2\sqrt{2}\,[g_2^2+g_3^2]^{1/2}} \Big\{ \varepsilon^*(p)\cdot\varepsilon^*(k)\ (p-k)\cdot\varepsilon(q)$$
$$+\ 2q\cdot\varepsilon^*(p)\ \varepsilon^*(k)\cdot\varepsilon(q)$$
$$-\ 2q\cdot\varepsilon^*(k)\ \varepsilon^*(p)\cdot\varepsilon(q)\Big\}$$

b/c $p\cdot\varepsilon^*(p) = k\cdot\varepsilon^*(k) = 0$

we can evaluate this for



$\longrightarrow \hat{3}$

If we average over the three polarizat states of $\varepsilon(q)$

So write $\quad \varepsilon^\mu(q) = (0, \hat{n}) \qquad q^\mu = (m_W, \vec{0})$

$\varepsilon^{*\mu}(p) = \frac{1}{\sqrt{2}}(0\ \ 1\ -i\ \ 0)$ R $\qquad \varepsilon^{*\mu}(k) = \frac{1}{\sqrt{2}}(0\ -1, -i, 0)$ R

$\frac{1}{\sqrt{2}}(0\ \ 1\ +i\ \ 0)$ L $\qquad \varepsilon^{*\mu}(k) = \frac{1}{\sqrt{2}}(0\ -1\ \ i\ \ 0)$ L

$\frac{1}{m_X}(p, 0, 0, E)$ 0 $\qquad \frac{1}{m_X}(p, 0\ 0\ -E)$ 0

$$2q\cdot\varepsilon^*(p) = \begin{cases} 0 & R \\ 0 & L \\ 2\frac{m_W}{m_X}p & 0 \end{cases} \qquad 2q\cdot\varepsilon^*(k) = \begin{cases} 0 & R \\ 0 & L \\ 2\frac{m_W}{m_X}p & 0 \end{cases}$$

$$\varepsilon^*(p)\cdot\varepsilon^*(k) = \begin{array}{ll} 1 & RR \\ 1 & LL \\ \dfrac{p^2+E^2}{m_X{}^2} & 00 \\ 0 & \text{all other cases.} \end{array}$$

$$(p-k)\cdot\varepsilon(q) = -2p\ \hat{3}\cdot\hat{n}$$

and if $\quad \vec{e}_+ = \dfrac{\hat{1}+i\hat{2}}{\sqrt{2}} \qquad \vec{e}_- = \dfrac{\hat{1}-i\hat{2}}{\sqrt{2}}$

$$\varepsilon^*(p)\cdot\varepsilon(q) = \begin{array}{ll} -\frac{1}{\sqrt{2}}\vec{e}_-\cdot\hat{n} & R \\ -\frac{1}{\sqrt{2}}\vec{e}_+\cdot\hat{n} & L \\ -\frac{E}{m_X}\hat{3}\cdot\hat{n} & 0 \end{array} \qquad \varepsilon^*(k)\cdot\varepsilon(q) = \begin{array}{l} \frac{1}{\sqrt{2}}\vec{e}_+\cdot\hat{n} \\ \frac{1}{\sqrt{2}}\vec{e}_-\cdot\hat{n} \\ \frac{E}{m_X}\hat{3}\cdot\hat{n} \end{array}$$

with these ingredients:

$$i\mathcal{M}(W\to RR) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}}\ 1\cdot(-2p\ \hat{3}\cdot\hat{n})$$

$$i\mathcal{M}(W\to RL) = \mathcal{M}(W\to LR) = 0$$

$$i\mathcal{M}(W\to LL) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}}\cdot 1\,[-2p(\hat{3}\cdot\hat{n})]$$

$$i\mathcal{M}(\mathcal{V}\to R0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \Big\{ 0 + 0 - 2\frac{m_W}{m_{\mathcal{I}}} p\cdot\left(-\tfrac{1}{\sqrt{2}}\vec{e}_-\cdot\hat{n}\right)$$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_{\mathcal{I}}}\, p\, \vec{e}_-\cdot\hat{n}$$

$$i\mathcal{M}(\mathcal{V}\to L0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_{\mathcal{I}}}\, p\, \vec{e}_+\cdot\hat{n}$$

$$i\mathcal{M}(\mathcal{V}\to R0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_{\mathcal{I}}}\, p\, \vec{e}_-\cdot\hat{n}$$

$$i\mathcal{M}(\mathcal{V}\to L0) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \sqrt{2}\frac{m_W}{m_{\mathcal{I}}}\, p\, \vec{e}_+\cdot\hat{n}$$

& finally

$$i\mathcal{M}(\mathcal{V}\to 00) = \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}} \Big\{ \frac{E^2+p^2}{m_{\mathcal{I}}^2}\cdot(-2p\,\hat{3}\cdot\hat{n}) + \left(2\frac{m_W}{m_{\mathcal{I}}}p\right)\left(\frac{E}{m_{\mathcal{I}}}\hat{3}\cdot\hat{n}\right) - \left(2\frac{m_W}{m_{\mathcal{I}}}p\right)\left(-\frac{E}{m_{\mathcal{I}}}\hat{3}\cdot\hat{n}\right)\Big\}$$

now $\quad E = \frac{m_{\mathcal{V}}}{2}$ or $m_{\mathcal{V}} = 2E \qquad p^2 = E^2 - m_{\mathcal{I}}^2$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}}\, p\cdot\hat{3}\cdot\hat{n}\,\frac{1}{m_{\mathcal{I}}^2}\Big\{ 8E^2 - 2E^2 - 2p^2\Big\}$$

$$= \frac{i g_3^2}{\sqrt{8(g_2^2+g_3^2)}}\, p\, \frac{\hat{3}\cdot\hat{n}}{m_{\mathcal{I}}^2}\Big\{ 4E^2 + 2m_{\mathcal{I}}^2\Big\}$$

in all

$$\sum_{\text{final spin}} |\mathcal{M}(\mathcal{V}^+ \to \mathcal{X}^{+2}\mathcal{X}^-)|^2 = \frac{g_3^4}{8(g_2^2+g_3^2)} \cdot \{(2p)^2\}$$

$$\cdot \Big\{ (\hat{3}\cdot\hat{n})^2 \cdot 2 + \Big(\frac{m_W}{m_{\mathcal{X}}} \frac{\vec{e}_-\cdot\hat{n}}{\sqrt{2}}\Big)^2 \cdot 2 + \Big(\frac{m_W}{m_{\mathcal{X}}} \frac{\vec{e}_+\cdot\hat{n}}{\sqrt{2}}\Big)^2 \cdot 2$$

$$+ (\hat{3}\cdot\hat{n})^2 \frac{1}{m_{\mathcal{X}}^4} (m_{\mathcal{V}}^2 + m_{\mathcal{X}}^2)^2 \Big\}$$

now avg over $\hat{n}$

$$\langle(\hat{3}\cdot\hat{n})^2\rangle = \frac{1}{3} = \langle(\hat{1}\cdot\hat{n})^2\rangle = \langle(\hat{2}\cdot\hat{n})^2\rangle$$

$$\langle(\vec{e}_-\cdot\hat{n})^2\rangle = \frac{1}{2}\left[(\hat{1}\cdot\hat{n})^2 + (\hat{2}\cdot\hat{n})^2\right] = \frac{1}{3} \quad \text{etc.}$$

$$= \frac{1}{3} \frac{g_3^4}{8(g_2^2+g_3^2)} 4p^2 \Big\{ 2 + \frac{4}{2}\frac{m_W^2}{m_{\mathcal{X}}^2} + \frac{(m_{\mathcal{V}}^2+m_{\mathcal{X}}^2)^2}{m_{\mathcal{X}}^4} \Big\}$$

$$= \frac{g_3^4}{24(g_2^2+g_3^2)} (m_{\mathcal{V}}^2 - 4m_{\mathcal{X}}^2) \Big\{ \frac{m_{\mathcal{V}}^4}{m_{\mathcal{X}}^4} + 4\frac{m_{\mathcal{V}}^2}{m_{\mathcal{X}}^2} + 3 \Big\}$$

$$= \frac{g_3^4}{24(g_2^2+g_3^2)} \cdot \frac{m_{\mathcal{V}}^6}{m_{\mathcal{X}}^4} \Big(1 - \frac{4m_{\mathcal{X}}^2}{m_{\mathcal{V}}^2}\Big) \Big\{ 1 + 4\frac{m_{\mathcal{X}}^2}{m_W^2} + 3\frac{m_{\mathcal{X}}^4}{m_W^4} \Big\}$$

then

$$\Gamma(\mathcal{V}^+ \to \mathcal{X}^{+2}\mathcal{X}^-) = \frac{1}{2m_{\mathcal{V}}} \frac{1}{8\pi} \langle \sum |\mathcal{M}|^2 \rangle \Big(1 - \frac{4m_{\mathcal{X}}^2}{m_{\mathcal{V}}^2}\Big)^{1/2}$$

so

$$\Gamma(\mathcal{W}^+ \to \mathcal{X}^{+2} \mathcal{X}^-) = \frac{\alpha_W}{96} m_W \cdot \frac{g_3^4}{g_2^4} \frac{m_W^4}{m_X^4} \cdot \left(1 - \frac{4 m_X^2}{m_W^2}\right)^{3/2} \cdot \left(1 + 4\frac{m_X^2}{m_W^2} + 3\frac{m_X^4}{m_W^4}\right)$$

finally (from p. 5)

$$\frac{m_W^2}{m_X^2} = \frac{g_2^2 + g_3^2}{\frac{1}{2} g_3^2}$$

so

$$\Gamma(\mathcal{W}^+ \to \mathcal{X}^{+2} \mathcal{X}^-) = \frac{\alpha_W}{24} m_W \left(1 + \frac{g_3^2}{g_2^2}\right)^2 \left(1 - \frac{4 m_X^2}{m_W^2}\right)^{3/2} \cdot \left(1 + 4\frac{m_X^2}{m_W^2} + 3\frac{m_X^4}{m_W^4}\right)$$

comparing to (p. 15)

$$\Gamma(\mathcal{W}^+ \to \ell^+ \nu) = \frac{\alpha_W}{12} m_W$$

we see that decay to fermions are numerically — but not parametrically — more important if $g_2 \gg g_3$

for more ~~interesting~~ analysis of this model, see

S. Dimopoulos + D.E. Kaplan

Phys. Lett. B531, 127 (2002) hep-ph/0201148

Phys. Lett. B534, 124 (2002) hep-ph/0202136

[extra-dimensional version]