Problem Set #9 - Solutions

1. (a) From Peskin (15.47),

$$-ig_1 A_{\mu_1}^a t^a \to (1+i\alpha_1^a t^a)(-ig_1 A_\mu^a t^a + \partial_\mu)(1-i\alpha_1^a t^a)$$
$$= (1+i\alpha_1^a t^a)(-ig_1 A_\mu^a t^a)(1-i\alpha_1^a t^a) - i(\partial_\mu \alpha_1^a) t^a$$

Then $D_\mu q_u \to \left(\partial_\mu + (1+i\alpha_1^a t^a)(-ig_1 A_{\mu_1}^a t^a)(1-i\alpha_1^a t^a) - i(\partial_\mu \alpha_1^a) t^a\right)(1+i\alpha_1^a t^a) q_u$

$$= (1+i\alpha_1^a t^a)\partial_\mu q_u + i(\partial_\mu \alpha_1^a) t^a q_u + (1+i\alpha_1^a t^a)(-ig_1 A_{\mu_1}^a t^a) q_u - i(\partial_\mu \alpha_1^a) t^a q_u$$
$$= (1+i\alpha_1^a t^a) D_\mu q_u$$

Similarly for $D_\mu q_t$

$$\Phi \to \Phi + (i\alpha_1^a t^a)\Phi + \Phi(-i\alpha_2^a t^a) = (1+i\alpha_1^a t^a)\Phi(1-i\alpha_2^a t^a)$$

$$D_\mu \Phi \to \partial_\mu\left((1+i\alpha_1^a t^a)\Phi(1-i\alpha_2^a t^a)\right)$$
$$+ \left[(1+i\alpha_1^a t^a)(-ig_1 A_{\mu_1}^a t^a)(1-i\alpha_1^a t^a) - i(\partial_\mu \alpha_1^a) t^a\right](1+i\alpha_1^a t^a)\Phi(1-i\alpha_2^a t^a)$$
$$+ (1+i\alpha_1^a t^a)\Phi(1-i\alpha_2^a t^a)\left[(1+i\alpha_2^a t^a)(ig_2 A_{\mu_2}^a t^a)(1-i\alpha_2^a t^a) + i\partial_\mu \alpha_2^a t^a\right]$$
$$= (1+i\alpha_1^a t^a)(D_\mu \Phi)(1-i\alpha_2^a t^a)$$

Thus covariant derivatives of fields transform same way as the fields themselves.

(b) Mass term is from $\mathrm{tr}((D_\mu \Phi)^\dagger D^\mu \Phi)$

$$\mathrm{tr}\left((D_\mu \langle\Phi\rangle)^\dagger D^\mu \langle\Phi\rangle\right) = V^2 \,\mathrm{tr}\left[(g_1 A_{\mu_1}^a t^a - g_2 A_{\mu_2}^a t^a)^2\right]$$
$$= V^2 \,\mathrm{tr}\left[g_1^2 A_{\mu_1}^a A_1^{\mu b} t^a t^b + g_2^2 A_{\mu_2}^a A_2^{\mu b} t^a t^b - 2 g_1 g_2 A_{\mu_1}^a A_2^{\mu b} t^a t^b\right]$$
$$= \frac{V^2}{2}(g_1 A_{\mu_1}^a - g_2 A_{\mu_2})^2 \qquad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$
$$= \frac{1}{2}(g_1^2+g_2^2)V^2\left(\frac{g_1}{\sqrt{g_1^2+g_2^2}} A_{\mu_1}^a - \frac{g_2}{\sqrt{g_1^2+g_2^2}} A_{\mu_2}^a\right)^2$$

So the linear combination $\frac{g_2}{\sqrt{g_1^2+g_2^2}} A_{\mu_1}^a + \frac{g_1}{\sqrt{g_1^2+g_2^2}} A_{\mu_2}^a \equiv G_\mu^a$ remains massless.

Note $\langle\Phi\rangle = V \cdot 1$ is invariant under a remaining $SU(3)$ group of transformation s.t. $\alpha_1^a = \alpha_2^a$ for all $a = 1, \dots, 8$, and G_μ^a is the corresponding massless mode

(c) To avoid confusion, we denote the massless eigenstates by G_μ^a, and the massive eigenstates by $A_\mu^a = \frac{g_1}{\sqrt{g_1^2+g_2^2}} A_{\mu_1}^a - \frac{g_2}{\sqrt{g_1^2+g_2^2}} A_{\mu_2}^a$

(b) shows A_μ^a has mass $m^2 = (g_1^2+g_2^2) V^2$

(d) Define $c = \cos\theta = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$, $s = \sin\theta = \frac{g_2}{\sqrt{g_1^2+g_2^2}}$

Then $\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \Rightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} A \\ G \end{pmatrix}$

Then $D_\mu q_u \ni (\partial_\mu - ig_1 \cdot s G_\mu^a t^a) q_u$

$D_\mu q_t \ni (\partial_\mu - ig_2 \cdot c\, G_\mu^a t^a) q_t$

$g_1 s = g_2 c = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}} \equiv g$

$\Downarrow$

$\frac{1}{g^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}$

(e) 1) First, for decay into gluons,

$(F_1^{\mu\nu a})^2 = (\partial_\mu A_\nu^{1a} - \partial_\nu A_\mu^{1a} + g_1 f^{abc} A_\mu^{1b} A_\nu^{1c})^2$

3-point vertex from $g_1 f^{abc} \partial_\mu A_\nu^{1a} A_1^{b\mu} A_1^{c\nu}$

$A_1 = cA + sG$, $A_2 = -sA + cG$

so $A \to GG$ vertex has $s^2 c\, g_1$ overall factor

Similarly from $(F_{\mu\nu}^{2a})^2$ we have $sc^2 g_2$ factor

Sum up, $s^2 c\, g_1 f^{abc} (\ldots) - sc^2 g_2 f^{abc} (\ldots) = 0$ since $s^2 c g_1 = sc^2 g_2$

Thus the heavy boson cannot decay into two gluons.

2) For decay into light quarks, $D_\mu q_u \ni (\partial_\mu - ig_1 c A_\mu^a t^a) q_u$

so we have vertex $\bar{q}_u (g_1 c \cancel{A}^a t^a) q_u$

$\Rightarrow iM = \bar{u}(p_1) \cdot (ig_1 c \gamma^\mu t^a) v(p_2) \cdot \epsilon_\mu(k)$.

For simplicity we treat the light quarks massless.

color average

$$\frac{1}{3}\cdot\frac{1}{8}\sum_{\substack{\text{colors}\\ \text{pols}}}|M|^2 = \frac{1}{24}\cdot\frac{g_1^4}{g_1^2+g_2^2}\, tr(t^a t^a)\, tr(\not p_1 \gamma^\mu \not p_2 \gamma^\nu)\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}\right)$$

polarization average

$$= \frac{1}{24}\cdot\frac{g_1^4}{g_1^2+g_2^2}\cdot 3\cdot\frac{4}{3}\cdot 4\cdot\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}\, p_1\cdot p_2\right)\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}\right)$$

$$= \frac{2}{3}\,\frac{g_1^4}{g_1^2+g_2^2}\left[2p_1\cdot p_2 + \frac{1}{m^2}\left(2(k\cdot p_1)(k\cdot p_2) - m^2 p_1\cdot p_2\right)\right]$$

In COM frame, $k^\mu = (m,0,0,0)$, $p_1^\mu = (\frac{m}{2},0,0,\frac{m}{2})$, $p_2^\mu = (\frac{m}{2},0,0,-\frac{m}{2})$

so $p_1\cdot p_2 = \frac{1}{2}M^2$, $k\cdot p_1 = k\cdot p_2 = \frac{m^2}{2}$.

Then $\frac{1}{3}\cdot\frac{1}{8}\sum|M|^2 = \frac{2}{3}\,\frac{g_1^4}{g_1^2+g_2^2}\,m^2$

$$\frac{d\Gamma}{d\Omega} = \frac{1}{32\pi^2}|M|^2\cdot\frac{|\vec{p}_1|}{m^2} \Rightarrow \Gamma = \frac{1}{16\pi}\frac{|M|^2}{m} = \frac{1}{24\pi}\cdot\frac{g_1^4}{g_1^2+g_2^2}\,m$$

3) Heavy quarks are also considered massless compared to m.

$$\Gamma = \frac{1}{24\pi}\,\frac{g_2^4}{g_1^2+g_2^2}\,m.$$

2. (a) $\frac{1}{2}\left(D_\mu\langle\phi^a\rangle\right)^2 = \frac{1}{2}\, g\epsilon^{abc} A_\mu^b v \delta^{3c} \cdot g\epsilon^{ade} A^{\mu d} v \delta^{3e}$

$= \frac{1}{2} g^2 v^2 \epsilon^{ab3} A_\mu^b \epsilon^{ad3} A^{\mu d}$

non-zero terms are when

$= \frac{1}{2} g^2 v^2 \left[(A_\mu^1)^2 + (A_\mu^2)^2 \right]$ $\quad a, b, d = 1, 2, 2$ and $2, 1, 1$

so A_μ^1, A_μ^2 obtain mass $m_w = gv$, while A_μ^3 stays massless.

(b) $D_\mu \psi = (\partial_\mu - ig A_\mu^a \cdot \frac{\sigma^a}{2})\psi$

$-ig A_\mu^3 \frac{\sigma^3}{2}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -i A_\mu^3 \begin{pmatrix}\frac{g}{2}\psi_1 \\ -\frac{g}{2}\psi_2\end{pmatrix}$ $\quad$ So $e = \frac{g}{2}$

(c) As $r \to \infty$ $\phi^a(r) \to v\hat{r}^a$, $\quad \phi^a\phi^a = v^2$

Note the potential for ϕ should be composed of ϕ^2 as ϕ is charged under $SU(2)$. Then $\langle\phi^2\rangle = $ const will be the minimum of the potential and our $\phi^a(r)$ satisfies this at $r\to\infty$

(d) Note $\partial_i \hat{r}^a = \partial_i\left(\frac{r^a}{r}\right) = \frac{r^2\delta^{ai} - r^a r^i}{r^3} = \frac{1}{r}\delta^{ai} - \frac{1}{r}\hat{r}^a\hat{r}^i$

$\phi^a = vf(r)\hat{r}^a$, and $f(r)\to 1$, $h(r) \to 1$ as $r\to\infty$.

Then as $r\to\infty$, $D_i\phi^a = \partial_i\phi^a + g\epsilon^{abc}A_i^b\phi^c$

$= v\,\partial_i\hat{r}^a - \frac{1}{r}\epsilon^{abc}\epsilon^{ibk}\hat{r}^k\phi^c$

$= v\,\partial_i\hat{r}^a - \frac{1}{r}\left(\delta^{ck}\delta^{ai} - \delta^{ci}\delta^{ak}\right)\hat{r}^k \cdot v\hat{r}^c$

$= \frac{v}{r}\left(\delta^{ai} - \hat{r}^a\hat{r}^i\right) - \left(\frac{v}{r}\delta^{ai}\hat{r}^k\hat{r}^k - \hat{r}^a\hat{r}^i\right) = 0$

So at $r\to\infty$, $D_i\phi^a$ is at most $\frac{1}{r^2}$

(e) $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g\epsilon^{abc}A_i^b A_j^c$

$= -\frac{1}{g}\left(\epsilon^{jak}\partial_i\left(\frac{h(r)\hat{r}^k}{r}\right) - \epsilon^{iak}\partial_j\left(\frac{h(r)\hat{r}^k}{r}\right) - \epsilon^{abc}\epsilon^{ibk}\epsilon^{jcl}\frac{h(r)\hat{r}^k}{r}\frac{h(r)\hat{r}^l}{r}\right)$

Denote $f^k = \frac{h(r)\hat{r}^k}{r}$ and note $\epsilon^{abc}\epsilon^{jcl} = \delta^{al}\delta^{kj} - \delta^{aj}\delta^{kl}$.

$$F_{ij}^{a} = -\frac{1}{g}\left(\epsilon^{jak}\partial_i f^k - \epsilon^{iak}\partial_j f^k + \delta^{aj}\epsilon^{ilk} f^l f^k\right)$$
$$= -\frac{1}{g}\left(\epsilon^{jak}\partial_i f^k - \epsilon^{iak}\partial_j f^k - \epsilon^{ijk} f^k f^a\right) \quad \text{as } \epsilon^{ilk} f^l f^k = 0.$$

Now compute $B^{ka} = -\frac{1}{2}\epsilon^{ijk} F_{ij}^{a}$

$$= -\frac{1}{2}\epsilon^{ijk}\left(-\frac{1}{g}\right)\left(\epsilon^{jal}\partial_i f^l - \epsilon^{ial}\partial_j f^l - \epsilon^{ijl} f^l f^a\right)$$
$$= \frac{1}{g}\left(\epsilon^{ijk}\epsilon^{jal}\partial_i f^l - \frac{1}{2}\epsilon^{ijk}\epsilon^{ijl} f^l f^a\right)$$
$$= \frac{1}{g}\left[\left(\delta^{ka}\delta^{li} - \delta^{ki}\delta^{al}\right)\partial_i f^l - f^k f^a\right]$$
$$= \frac{1}{g}\left(\partial_i f^i \, \delta^{ka} - \partial_k f^a - f^k f^a\right)$$

$f^k = \frac{h(r)}{r}\hat{r}^k$, so $\partial_i f^i = \frac{1}{r^2}\partial_r(rh) = \frac{1}{r^2}h + \frac{1}{r}\partial_r h$

$$\partial_k f^a = \partial_k\left(\frac{h(r)}{r^2} r^a\right) = \frac{h}{r^2}\delta^{ka} + \frac{r^a r^k}{r}\partial_r\left(\frac{h}{r^2}\right)$$
$$= \frac{h}{r^2}\delta^{ka} + \frac{r^a r^k}{r^3}\partial_r h - 2\frac{r^a r^k}{r^4}h$$

Then, $$B^{ka} = \frac{1}{g}\left[\left(\frac{1}{r^2}h + \frac{1}{r}\partial_r h\right)\delta^{ka} - \frac{h}{r^2}\delta^{ka} - \frac{r^a r^k}{r^3}\partial_r h + 2\frac{r^a r^k}{r^4}h - \frac{h^2}{r^2}\hat{r}^a\hat{r}^k\right]$$
$$= \frac{1}{g}\left[\frac{1}{r}\partial_r h\left(\delta^{ka} - \hat{r}^k\hat{r}^a\right) + \left(2h - h^2\right)\frac{\hat{r}^a\hat{r}^k}{r^2}\right]$$

(f) Assume $\partial_r h \to 0$ as $r \to \infty$. $h \to 1$ as $r \to \infty$

$B^{ka} = \frac{\hat{r}^k\hat{r}^a}{g r^2} \Rightarrow$ magnetic charge $\frac{1}{g}$

(g) $e = \frac{g}{2} \Rightarrow$ magnetic charge $\frac{1}{g} = \frac{1}{2e}$

Note the Dirac quantization condition : $2 q_m \cdot q_e \in \mathbb{Z}$

here we have $2 \cdot \frac{1}{2e} \cdot e = 1$.