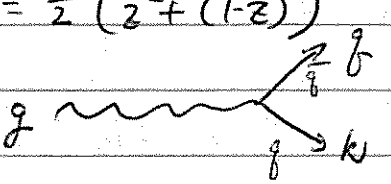
Problem Set #8 - Solutions

1. First consider $P_{q\leftarrow g}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right)$

The relevant diagram is



$iM = \bar{u}(k)(-ig\gamma_\mu t^a)v(q)\epsilon_T^\mu(p)$.

Consider the case that outgoing q is left-handed

then $u_L(k) = \sqrt{2k^0}\begin{pmatrix}\xi(k)\\0\end{pmatrix}$ (note $m=0$) $\quad \xi(k) = \begin{pmatrix} p_\perp/2(1-z)p \\ 1\end{pmatrix}$

$v_L(q) = \sqrt{2q^0}\begin{pmatrix}\eta(k)\\0\end{pmatrix}$ $\quad \eta(k) = \begin{pmatrix} -p_\perp/2zp \\ 1\end{pmatrix}$

$\gamma^0\gamma_\mu = \begin{pmatrix}\bar{\sigma}_\mu & 0\\ 0 & \sigma_\mu\end{pmatrix}$ and $k^0 = (1-z)p$, $q^0 = zp$.

Then $\bar{u}_L(k)\gamma_\mu v_L(q) = \sqrt{2(1-z)p}\,\sqrt{2zp}\,\xi^\dagger(k)\,\bar{\sigma}_\mu\,\eta(q)$

and $iM = ig\sqrt{2(1-z)p}\,\sqrt{2zp}\,\xi^\dagger(k)\,\sigma^i\eta(q)\,\epsilon_T^i(p)\,t^a$

Note $\epsilon_T^0 = 0$, and $\epsilon_L^i(p) = \frac{1}{\sqrt{2}}(1,-i,0)$, $\epsilon_R^i(P) = \frac{1}{\sqrt{2}}(1,i,0)$

So we get $iM(g_L\to q_L\bar{q}_R) = -ig\frac{\sqrt{2z(1-z)}}{z}p_\perp t^a$

$$iM(g_R\to q_L\bar{q}_R) = +ig\frac{\sqrt{2z(1-z)}}{1-z}p_\perp t^a$$

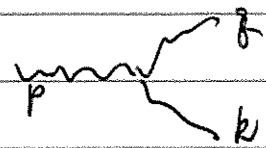
Note $M(g_L\to q_L\bar{q}_R) = M(g_R\to q_R\bar{q}_L)$

$$M(g_R\to q_L\bar{q}_R) = M(g_L\to q_R\bar{q}_L)$$

Note $\frac{1}{8}\,tr(t^at^a) = \frac{1}{8}\cdot 3\cdot\frac{4}{3} = \frac{1}{2}$

Then $\frac{1}{2}\cdot\frac{1}{8}\sum_{\substack{pol\\color}}|M|^2 = \frac{2g^2p_\perp^2}{z(1-z)}\cdot\frac{1}{2}\left(z^2+(1-z)^2\right)$ which implies $P_{q\leftarrow g}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right)$

Now, for $P_{g\leftarrow g}$, consider



$p = (p,0,0,p)$, $q = (zp, p_\perp, 0, zp)$, $k = ((1-z)p, -p_\perp, 0, (1-z)p)$.

$\epsilon_L^i(p) = \frac{1}{\sqrt{2}}(1, -i, 0)$, $\epsilon_R^i(p) = \frac{1}{\sqrt{2}}(1, i, 0)$

$\epsilon_L^i(q) = \frac{1}{\sqrt{2}}\left(1, -i, -\frac{p_\perp}{zp}\right)$, $\epsilon_R^i(q) = \frac{1}{\sqrt{2}}\left(1, i, -\frac{p_\perp}{zp}\right)$

$\epsilon_L^i(k) = \frac{1}{\sqrt{2}}\left(1, -i, \frac{p_\perp}{(1-z)p}\right)$, $\epsilon_R^i(k) = \frac{1}{\sqrt{2}}\left(1, i, \frac{p_\perp}{(1-z)p}\right)$,

Note

$$
\begin{aligned}
iM = g f^{abc}\Big[&(\epsilon^*(q)\cdot\epsilon(p))\,((p+q)\cdot\epsilon^*(k)) \\
&+ (\epsilon^*(q)\cdot\epsilon^*(k))\,((k-q)\cdot\epsilon(p)) \\
&- (\epsilon^*(k)\cdot\epsilon(p))\,((p+k)\cdot\epsilon^*(q))\Big]
\end{aligned}
$$

Apply proper ϵ's, we get

$g_L \to g_L g_L$: $iM = g f^{abc}\frac{\sqrt{2}\,p_\perp}{z(1-z)}$

$g_L \to g_L(q)\, g_R(k)$: $iM = g f^{abc}\frac{\sqrt{2}\,z\,p_\perp}{1-z}$

$g_L \to g_R g_L$: $iM = g f^{abc}\frac{\sqrt{2}\,(1-z)p_\perp}{z}$

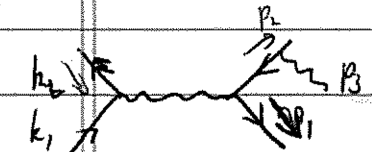
$g_R \to g_R g_R$: $iM = 0$, as it should be.

Now, with parity invariance, and $f^{abc}f^{abc} = 24$.

$$
\frac{1}{2}\cdot\frac{1}{8}\sum_{\substack{\text{spin}\\\text{color}}}|M|^2 = \frac{2g^2p_\perp^2}{z(1-z)}\cdot\frac{1}{2}\cdot\frac{1}{8}\cdot 24\cdot 2\left(\frac{1}{z(1-z)} + \frac{z^3}{1-z} + \frac{(1-z)^3}{z}\right)
$$

$$
= \frac{2g^2p_\perp^2}{z(1-z)}\cdot\underbrace{3\cdot\frac{z^4+(1-z)^4+1}{z(1-z)}}_{P_{g\leftarrow g}(z)}
$$

Note, fixing the singularity at $z=1$, you can obtain Eq. (17.130) in Peskin P590

2. (a) 

$$iM = \bar{v}(k_2)\,(-ie\gamma^\mu)\,u(k_1)\cdot\frac{-i}{Q^2}\times$$

$$\Big[\ \bar{u}(p_1)\,(-ieQ_f\gamma_\mu)\cdot\frac{i(\not p_2+\not p_3)}{(p_2+p_3)^2}\,(-ig\gamma^\nu t^a)\,v(p_2)\,\epsilon_\nu^*(p_3)$$

note the "−" sign.
$$\ominus\,\bar{u}(p_1)\,(-ig\gamma^\nu t^a)\,\frac{i(\not p_1+\not p_3)}{(p_1+p_3)^2}\,(-ieQ_f\gamma_\mu)\,v(p_2)\,\epsilon_\nu^*(p_3)\Big]$$

$$= i\,\frac{e^2Q_f g t^a}{Q^2}\,\bar{v}(k_2)\gamma^\mu u(k_1)\,\bar{u}(p_1)\Big(\gamma_\mu\cdot\frac{1}{\not p_2+\not p_3}\gamma_\nu-\gamma_\nu\frac{1}{\not p_1+\not p_3}\gamma_\mu\Big)v(p_2)\,\epsilon_\nu^*(p_3).$$

Note w/ $\epsilon^{\nu*}(p_2)\to p_3^\nu$, we get

$$\bar{u}(p_1)\Big(\gamma_\mu\frac{1}{\not p_2+\not p_3}\not p_3-\not p_3\frac{1}{\not p_1+\not p_2}\gamma_\mu\Big)v(p_2)\qquad \text{note } \not p_3=\not p_2+\not p_3-\not p_2$$
$$=\not p_1+\not p_3-\not p_1$$

$$=\bar{u}(p_1)\Big(\gamma_\mu-\gamma_\mu\frac{1}{\not p_2+\not p_3}\not p_2-\gamma_\mu+\not p_1\cdot\frac{1}{\not p_1+\not p_3}\gamma_\mu\Big)v(p_2)$$

$$=0\quad \text{as } \bar{u}(p_1)\not p_1=0,\ \not p_2 v(p_2)=0.$$

(b) $x_1+x_2+x_3=\dfrac{2(p_1+p_2+p_3)Q}{Q^2}=2$

In CM frame, $Q^\mu=(Q,0,0,0)$ and the maximum E of a particle is $\frac{Q}{2}$.
(two particles going in parallel, the third anti-parallel).

$\Rightarrow\ p_i^0\in[0,\frac{Q}{2}]$, $\quad x_i\in\dfrac{2p_i^0Q}{Q^2}\in[0,1]$.

Again $x_i=1$ means particle i going in one direction and the other two particles going opposite to i.

(c) In CM frame, $x_i = \frac{2E_i Q}{Q^2} = \frac{2E_i}{Q}$.

Note $p_1 + p_2 + p_3 = Q$

$\Rightarrow (p_1+p_2)^2 = (Q-p_3)^2$

$\Rightarrow p_1^2 + 2p_1 \cdot p_2 + p_2^2 = Q^2 - 2p_3 \cdot Q + p_3^2$ and $p_1^2 = p_2^2 = p_3^2 = 0$.

so $2(E_1 E_2 - E_1 E_2 \cos\theta_{12}) = Q^2 - Q^2 x_3$

Divide by Q^2, $\frac{1}{2} x_1 x_2 (1-\cos\theta_{12}) = 1 - x_3$.

(d) $\int d\Pi_3 = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3p_2}{(2\pi)^3} \cdot \frac{1}{2E_2} \cdot \frac{d^3p_3}{(2\pi)^3} \cdot \frac{1}{2E_3} (2\pi)^4 \delta^{(4)}(Q-p_1-p_2-p_3)$.

$$= \frac{1}{(2\pi)^5} \int d^3p_1\, d^3p_2 \cdot \frac{1}{8E_1E_2E_3} \delta(Q-E_1-E_2-E_3)$$

$$= \frac{1}{(2\pi)^5} \cdot 4\pi \cdot 2\pi \int dE_1\, dE_2\, d\cos\theta_{12} \frac{E_1^2 E_2^2}{8E_1E_2E_3} \cdot \frac{E_3}{E_1E_2} \delta\left(\cos\theta_{12} - \frac{E_3^2 - E_1^2 - E_2^2}{2E_1E_2}\right).$$

$$= \frac{1}{32\pi^3} \int dE_1\, dE_2 = \frac{Q^2}{128\pi^3} \int dx_1\, dx_2 \quad \text{as } E_i = \frac{Q}{2} x_i$$

(e) $0 \le x_i \le 1$ so $0 \le x_1, x_2 \le 1$, and $0 \le x_3 \le 1$ and $x_1 + x_2 + x_3 = 2$ implies $x_1 + x_2 \ge 1$.

(f) $\frac{1}{4} \mathrm{tr}\left(\bar{v}(k_2)\gamma^\mu u(k_1)\bar{u}(k_1)\gamma^\nu v(k_2)\right)$

$$= \frac{1}{4}\mathrm{tr}(\not{k}_2 \gamma^\mu \not{k}_1 \gamma^\nu) = k_2^\mu k_1^\nu + k_1^\mu k_2^\nu - g^{\mu\nu} k_1 \cdot k_2$$

Note $k_1^\mu = \frac{Q}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and $\vec{k_2} = -\vec{k_1}$

and $\int d\cos\theta \cos\theta = 0 \qquad \int d\phi \cos\phi = \int d\phi \sin\phi = 0$

$\int d\phi \cos^2\phi = \int d\phi \sin^2\phi = \pi, \qquad \int d\phi \sin\phi\cos\phi = 0$

$\int d\cos\theta \cos^2\theta = \frac{2}{3}, \qquad \int d\cos\theta \sin^2\theta = \frac{4}{3} \qquad \int d\cos\theta \sin\theta\cos\theta = 0$

Then after averaging over $d\Omega$ (i.e. $\frac{1}{4\pi}\int d\Omega$),

we get $k_1^\mu k_2^\nu = \left(\frac{Q}{2}\right)^2 \begin{pmatrix} 1 & & & \\ & -1/3 & & \\ & & -1/3 & \\ & & & -1/3 \end{pmatrix}$ and $k_1 \cdot k_2 = \frac{Q^2}{2}$

so $k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - g^{\mu\nu} k_1 \cdot k_2 = \frac{Q^2}{2}\begin{pmatrix} 0 & & & \\ & 2/3 & & \\ & & 2/3 & \\ & & & 2/3 \end{pmatrix} = -\frac{Q^2}{3}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)$ for $Q^\mu = (Q, \vec{0})$

(g) Using (f), $\frac{1}{4}\sum_{\substack{\text{spins} \\ \text{colors}}} |M(e^+e^- \to q\bar{q})|^2 = \underset{\substack{\uparrow \\ \text{color} \\ \text{factor}}}{3} (e^2 Q_f)^2 \left(-\frac{Q^2}{3}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)\right) \cdot \frac{1}{Q^4} \cdot \underset{\substack{\uparrow \\ \text{sum over spins,} \\ \text{not averaged}}}{4} \cdot \left(-\frac{Q^2}{3}\left(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}\right)\right)$

$= \frac{4e^4 Q_f^2}{3}(4-2+1) = 4e^4 Q_f^2$

Then $\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 Q^2} = \frac{e^4}{16\pi^2}\frac{Q_f^2}{Q^2}$ Then $\sigma = \frac{e^4}{4\pi} \cdot \frac{Q_f^2}{Q^2} = \frac{4\pi\alpha^2}{3Q^2}(3Q_f^2)$

(h) $\frac{1}{4}|M|^2 = \frac{g^2 e^4 Q_f^2}{Q^4} \operatorname{tr}(t^a t^a)\left[-\frac{Q^2}{3}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)\right] \times$

$\operatorname{tr}\left[\bar{u}(p_1)\left(\gamma_\mu \frac{1}{\not{p}_2 + \not{p}_3}\gamma_\rho - \gamma_\rho \frac{1}{\not{p}_1 + \not{p}_3}\gamma_\mu\right) v(p_2)\, \bar{v}(p_2)\left(\gamma_\sigma \frac{1}{\not{p}_2+\not{p}_3}\gamma_\nu - \gamma_\nu \frac{1}{\not{p}_1+\not{p}_3}\gamma_\sigma\right) u(p_1)\right] \times (-g^{\rho\sigma})$

Note $Q^\mu = (p_1 + p_2 + p_3)^\mu$

and $\bar{u}(p_1)\left(\not{Q}\frac{1}{\not{p}_2+\not{p}_3}\gamma_\rho - \gamma_\rho \frac{1}{\not{p}_1+\not{p}_3}\not{Q}\right) v(p_2) = \bar{u}(p_1)(\gamma_\rho - \gamma_\rho) v(p_2) = 0$

so $\frac{Q^\mu Q^\nu}{Q^2}$ has no contribution

and $g^{\mu\nu} g^{\rho\sigma} \operatorname{tr}\left[\not{p}_1\left(\gamma_\mu \frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\gamma_\rho - \gamma_\rho \frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma_\mu\right) \cdot \not{p}_2\left(\gamma_\sigma \frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\gamma_\nu - \gamma_\nu \frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma_\sigma\right)\right]$

$= \operatorname{tr}\left[4\not{p}_1 \frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\not{p}_2 \frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2} + 4\not{p}_1 \frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\not{p}_2 \frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2} + 8\not{p}_1\not{p}_2 \frac{(p_1+p_3)\cdot(p_2+p_3)}{(p_1+p_3)^2(p_2+p_3)^2} \times 2\right]$

from $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

If we further use $\not{p}_2\not{p}_2 = p_2^2 = 0$ etc

above $= 4\operatorname{tr}(\not{p}_1\not{p}_3\not{p}_2\not{p}_3) \times \left(\frac{1}{(p_2+p_3)^4} + \frac{1}{(p_1+p_3)^4}\right) + 64\frac{(p_1+p_3)\cdot(p_2+p_3)(p_1 \cdot p_2)}{(p_1+p_3)^2(p_2+p_3)^2}$

$= 32(p_1 \cdot p_3)(p_2 \cdot p_3)\left(\frac{1}{(p_2+p_3)^4} + \frac{1}{(p_1+p_3)^4}\right) + 64(p_1 \cdot p_2)\frac{p_1\cdot p_2 + p_2 \cdot p_3 + p_1 \cdot p_3}{(p_1+p_3)^2(p_2+p_3)^2}$

Note $2p_i \cdot p_j = (p_i + p_j)^2$ and $2(p_1\cdot p_2 + p_2 \cdot p_3 + p_1 \cdot p_3) = (p_1+p_2+p_3)^2 = Q^2$

above $= \frac{8}{(p_2+p_3)^2(p_1+p_3)^2}\left((p_1+p_3)^4 + (p_2+p_3)^4 + 2(p_1+p_2)^2 Q^2\right)$

$= \frac{8}{(1-x_1)(1-x_2)}\left((1-x_2)^2 + (1-x_1)^2 + 2(1-x_3)\right) = \frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}$ (note $x_3 = 2 - x_1 - x_2$)

Then overall, $\frac{1}{4}\sum|M|^2 = \frac{g^2 e^4 Q_f^2}{Q^4}\cdot\frac{4}{3}\cdot 3\,\frac{Q^2}{3}\,\frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}$

$$= \frac{32}{3}\cdot\frac{1}{Q^2}\,g^2e^4Q_f^2\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

Then $\frac{d\sigma}{dx_1\,dx_2} = \frac{1}{2Q^2}\,\frac{Q^2}{128\pi^3}\,\frac{1}{4}\sum|M|^2$

$$= \frac{g^2e^4Q_f^2}{24\pi^3Q^2}\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} = \frac{4\pi\alpha^2}{3Q^2}\,3Q_f^2\cdot\frac{2\alpha_s}{3\pi}\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

(i) x_1 : quarks $\rightarrow 1-z$

x_2 : antiquark

x_3 : gluon $\rightarrow z$

$$\frac{d\sigma}{dx_1dx_2} = \sigma_0\cdot\frac{2\alpha_s}{3\pi}\cdot\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} \xrightarrow[x_2\to 1]{} \sigma_0\cdot\frac{\alpha_s}{2\pi}\cdot\frac{4}{3}\,\frac{x_1^2+1}{1-x_1}\cdot\frac{1}{1-x_2}$$

The divergence $\frac{1}{1-x_2}$ corresponds to $\frac{1}{p_\perp^2}$

then $\frac{dx_2}{1-x_2} = -\frac{dp_\perp^2}{p_\perp^2}$, and $1-x_1 = z$

so we have $$\frac{d\sigma}{dz} = \sigma_0\cdot\frac{\alpha_s}{2\pi}\cdot\underbrace{\frac{4}{3}\cdot\frac{1+(1-z)^2}{z}}_{P_{g\leftarrow q}(z)}\,\frac{dp_\perp^2}{p_\perp^2}$$