Problem Set #7 – Solutions

1. (a) $\psi_i \to \gamma^5 \psi_i \Rightarrow \bar{\psi}_i \to \psi_i^\dagger \gamma^{5\dagger} \gamma^0 = -\bar{\psi}_i \gamma^5$

$\gamma^\mu \psi_i \to \gamma^\mu \gamma^5 \psi_i = -\gamma^5 \gamma^\mu \psi_i$

Then $\bar{\psi}_i i\not{\partial} \psi_i \longrightarrow (-\bar{\psi}_i \gamma^5)(-\gamma^5 i\not{\partial}\psi_i) = \bar{\psi}_i i\not{\partial}\psi_i$

$\bar{\psi}_i \psi_i \longrightarrow -\bar{\psi}_i \psi_i \Rightarrow$ mass term not invariant under $\psi_i \to \gamma^5\psi_i$, so mass term is forbidden by this symmetry.

Therefore $\mathcal{L}$ is invariant under $\psi_i \to \gamma^5 \psi_i$

(b) $S \supset \int \underbrace{d^2x}_{-2} \ \bar{\psi}_i i \underbrace{\not{\partial}}_{1} \psi_i$

$[\psi_i] = \frac{1}{2} \Rightarrow [(\bar{\psi}_i\psi_i)^2] = 2$ so $[g] = 0$

(c) Integrate out σ in

$$\int D\bar{\psi}\, D\psi\, D\sigma \ \exp\left\{ i\int d^2x \left(\bar{\psi}_i i\not{\partial}\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2 \right)\right\}$$

$$= \int D\bar{\psi}\, D\psi \ \exp\left(i\int d^2x \left(\bar{\psi}_i i\not{\partial}\psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2\right)\right) \int D\sigma \exp\left(-\frac{i}{2g^2}\int d^2x\,(\sigma+g^2\bar{\psi}_i\psi_i)^2\right)$$

$$= C \cdot \int D\bar{\psi}\, D\psi \ \exp\left(i\int d^2x\, \mathcal{L}\right)$$

(d) From (c), integrate out ψ_i, $\bar{\psi}_i$

$$\int D\sigma\, D\psi_i\, D\bar{\psi}_i \ \exp\left[i\int d^2x\ \bar{\psi}_i (i\not{\partial}-\sigma)\psi_i - \frac{1}{2g^2}\sigma^2\right]$$

$$= \int D\sigma \ \exp\left(-\frac{i}{2g^2}\int d^2x\, \sigma^2\right) \left[\det(i\not{\partial}-\sigma)\right]^N \qquad i = 1, \dots, N$$

$\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, $\gamma^5 = \gamma^0\gamma^1 = \sigma^3$, $(\gamma^5)^2 = 1$.

$\partial^2 = \partial_0^2 - \partial_1^2$

$$\det(i\not{\partial}-\sigma) = \det\begin{pmatrix} -\sigma & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & -\sigma \end{pmatrix} = \det\left(\sigma^2 - (-\partial_0^2 + \partial_1^2)\right) = \det(\sigma^2 + \partial^2)$$

$$= \exp\left[\mathrm{tr}\log(\sigma^2+\partial^2)\right]$$

$$\mathrm{tr}\log(\sigma^2+\partial^2) = VT\left[\int \frac{d^d p}{(2\pi)^d}\log(\sigma^2-p^2)\right]$$

$$\overset{p^0=ip_E^0}{=} iVT\int\frac{d^d p_E}{(2\pi)^d}\log(\sigma^2+p_E^2)$$

$$\left.\frac{\partial}{\partial\alpha}x^{-\alpha}\right|_{\alpha=0} = \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=0} e^{-\alpha\log x} = -\log x.$$

$$\mathrm{tr}\log(\sigma^2+\partial^2) = -iVT\left.\left(\frac{\partial}{\partial\alpha}\int\frac{d^d p_E}{(2\pi)^d}\cdot\frac{1}{(p_E^2+\sigma^2)^\alpha}\right)\right|_{\alpha=0}$$

$$= -iVT\,\frac{\partial}{\partial\alpha}\left.\left(\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\times\frac{1}{(\sigma^2)^{\alpha-d/2}}\right)\right|_{\alpha=0}$$

use $\Gamma(\alpha)\to 1/\alpha$ as $\alpha\to 0$

$$= -iVT\,\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}(\sigma^2)^{d/2}$$

$\Gamma(x)=(x-1)\Gamma(x-1)$

$$= iVT\,\frac{\sigma^2}{4\pi}\cdot\frac{\Gamma(1-\frac{d}{2})}{d/2}\left(\frac{4\pi}{\sigma^2}\right)^{1-\frac{d}{2}}$$

$\epsilon = 1-\frac{d}{2}$

$$= VT\frac{i\sigma^2}{4\pi}\left(\frac{1}{\epsilon}-\gamma+\log 4\pi+1-\log(\sigma^2)\right)$$

Subtrait the infinity by a δ_g counter-term in $\overline{MS}$ scheme, the det becomes

$$\det(i\not{\partial}-\sigma) = \exp\left[-iVT\frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{M^2}-1\right)\right]$$

(e) $\int D\sigma \exp\left[i\int d^2x \left(-\frac{\sigma^2}{2g^2} - N\frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{M^2}-1\right)\right)\right]$

That is, $V_{eff}(\sigma) = \frac{\sigma^2}{2g^2} + N\cdot\frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{M^2}-1\right)$

$V'(\sigma) = \frac{\sigma}{g^2} + \frac{N}{2\pi}\sigma\left(\log\frac{\sigma^2}{M^2}-1\right) + \frac{N}{2\pi}\sigma$

$V'(\sigma) = 0$ when $\sigma = \pm M\exp\left(-\frac{\pi}{g^2N}\right)$ or $\sigma = 0$

The minimum is at $\langle\sigma\rangle = \pm M\exp\left(-\frac{\pi}{g^2N}\right)$

Note for small g, $\langle\sigma\rangle \to 0$.

(f) Note the hierarchy formula,

$$\Lambda_{\text{confinement}} = \Lambda\exp\left(-\frac{8\pi^2}{\frac{11}{3}C_2(G)}\cdot\frac{1}{g^2(\Lambda)}\right)$$

Note the same form of exponent $\propto -\frac{1}{g^2}$

which implies this model is also asymptotically free.

2. (a) $H = \frac{1}{2}(-\partial^2 + m^2)$ (Euclidean time).

$$\int_0^\infty dT \langle x_f | e^{-HT} | x_i \rangle = \int_0^\infty dT \int \frac{d^4k}{(2\pi)^4} \langle x_f | k \rangle \langle k | e^{-HT} | x_i \rangle$$

$$= \int_0^\infty dT \int \frac{d^4k}{(2\pi)^4} \langle x_f | k \rangle e^{-\frac{1}{2}(k^2+m^2)T} \langle k | x_i \rangle$$

$$= 2 \int \frac{d^4k}{(2\pi)^4} \langle x_f | k \rangle \frac{1}{k^2+m^2} \langle k | x_i \rangle = \langle x_f | \frac{2}{k^2+m^2} | x_i \rangle$$

(b) In momentum space (Euclidean time).

$$-(\partial_\mu - igA_\mu)^2 + m^2 \longrightarrow -(-ik_\mu - igA_\mu)^2 + m^2 = (k_\mu + gA_\mu)^2 + m^2$$

Then, $\langle x_j | e^{-H\Delta t} | x_{j-1} \rangle = \int \frac{d^4k_E}{(2\pi)^4} e^{-\Delta t (k^2 + 2gk\cdot A + g^2A^2 + m^2)} \overbrace{\langle x_j | k \rangle}^{e^{ikx_j}} \overbrace{\langle k | x_{j-1} \rangle}^{e^{-ikx_{j-1}}}$

$$= \int \frac{d^4k_E}{(2\pi)^4} \exp\left[-\Delta t \left(k_\mu + gA_\mu + i\frac{\Delta x_\mu}{2\Delta t}\right)^2 + \Delta t \left(gA_\mu + i\frac{\Delta x_\mu}{2\Delta t}\right)^2 - \Delta t (g^2A^2 + m^2)\right]$$

$$\propto \exp\left\{-\Delta t \left[\left(\frac{1}{2}\frac{\Delta x}{\Delta t}\right)^2 - ig \frac{d\vec{x}}{dt}\cdot\vec{A} + m^2\right]\right\}$$

$\downarrow$

gives $\exp(ig\int d\vec{x}\cdot\vec{A})$ additional contribution

(c) See Peskin P492–493.

Replace by $P\left\{\exp\left[ig\int ds \frac{dx^\mu}{ds} A_\mu^a(x(s))\, t^a\right]\right\}$, P : path-ordering.

(d) $- (\not{D}+m)(\not{D}-m) = -\not{D}^2 + m^2$

$= -\gamma^\mu \gamma^\nu D_\mu D_\nu + m^2$

$= \left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} - \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) D_\mu D_\nu + m^2$

$= -D^2 + 2i \frac{i}{4}[\gamma^\mu, \gamma^\nu] D_\mu D_\nu + m^2$

$= -D^2 + m^2 + 2i\left(\frac{1}{2} \cdot \frac{i}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu]\right)$

$= -D^2 + m^2 + 2i \cdot \left(\frac{1}{2} S^{\mu\nu} F_{\mu\nu}\right)$

$= -D^2 + m^2 + 2i \cdot \left(\frac{1}{2} S_{mn} F^{mna} t^a\right)$

(e)

$P\left[\exp\left\{ig \int dx^\mu \left(A_\mu^a t^a - S_{mn} F^{mna} t^a\right)\right\}\right]$