Problem Set 6 - Solution.

(a) $[t^a, t^b] = i f^{abc} t^c$

Then $\mathrm{tr}(t^a t^b t^a t^b) = \mathrm{tr}(t^a t^a t^b t^b) - i f^{abc}\, \mathrm{tr}(t^a t^c t^b)$

Note $f^{abc} t^a t^c t^b = \frac{1}{2}\left(f^{abc} t^a t^c t^b + f^{acb} t^a t^b t^c\right)$

$$= \frac{1}{2} f^{abc} t^a [t^c, t^b] = \frac{i}{2} f^{abc} f^{cbd} t^a t^d$$

$$= -\frac{i}{2} C_2(G)\, t^a t^a$$ (use f^{abc} totally anti-symmetric, and Peskin Eqn. (15.93))

Then $\mathrm{tr}(t^a t^b t^a t^b) = C_2(r)^2 d(r) - \frac{1}{2} C_2(G) C_2(r) d(r)$

$$= d(r) \cdot C_2(r) \left(C_2(r) - \frac{1}{2} C_2(G)\right)$$

For $SU(3)$ and $r = 3$, this is $= 3 \cdot C_2(3)\left(C_2(3) - \frac{1}{2} C_2(G)\right) = -\frac{2}{3}$.

(b) In $qq \to qq$ and $q\bar{q} \to q\bar{q}$ interactions, all vertices have spinor structures

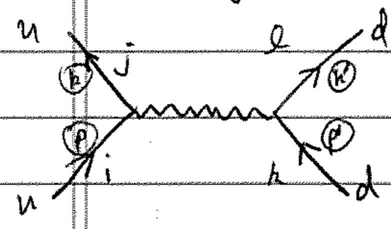
$\bar{u}\gamma^\mu u$, $\bar{u}\gamma^\mu v$, $\bar{v}\gamma^\mu u$, $\bar{v}\gamma^\mu v$

Since $\gamma^0\gamma^\mu \frac{1\pm\gamma^5}{2} = \frac{1\pm\gamma^5}{2}\gamma^0\gamma^\mu$, each pair of fermions should have same vertices.

In terms of $(+-)$ notation, each vertex has $(+-)$ or $(-+)$.

So overall two $+$ and two $-$

(c) Only one diagram (t-channel)



$$= \bar{u}(k)\, i g \gamma^\mu t^a_{ij} u(p) \frac{-i g_{\mu\nu}}{(p-k)^2} \bar{u}(k')\, i g \gamma^\nu t^a_{kl} u(p')$$

$$= \frac{i g^2}{t} \bar{u}(k)\gamma^\mu u(p)\, \bar{u}(k')\gamma_\mu u(p')\, t^a_{ij} t^a_{kl}$$

In the COM frame, we choose

$$p = E(1, 0, 0, 1), \quad p' = E(1, 0, 0, -1)$$
$$k = E(1, \sin\theta, 0, \cos\theta), \quad k' = E(1, -\sin\theta, 0, -\cos\theta)$$

Note $s = 4E^2$, $t = -4E^2 \sin^2\frac{\theta}{2}$, $u = -4E^2\cos^2\frac{\theta}{2}$

We compute $\bar{u}\gamma^\mu u$ for each helicity combination.
Be careful w/ moving in $+\hat{z}$ vs. $-\hat{z}$ direction.

Then $u_+(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u_-(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $u_+(p') = \sqrt{2E}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $u_-(p') = \sqrt{2E}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$u_+(k) = \sqrt{2E}\begin{pmatrix}0\\0\\c\\s\end{pmatrix}$, $u_-(k) = \sqrt{2E}\begin{pmatrix}-s\\c\\0\\0\end{pmatrix}$, $u_+(k') = \sqrt{2E}\begin{pmatrix}0\\0\\-s\\c\end{pmatrix}$, $u_-(k') = \sqrt{2E}\begin{pmatrix}-c\\-s\\0\\0\end{pmatrix}$

where $c = \cos\frac{\theta}{2}$, $s = \sin\frac{\theta}{2}$.

Then $\bar{u}_+(k)\gamma^\mu u_+(p) = 2E\,(c \;\; s \;\; is \;\; c)$
$\bar{u}_+(k')\gamma^\mu u_+(p') = 2E\,(c \;\; -s \;\; is \;\; -c)$
$\bar{u}_-(k)\gamma^\mu u_-(p) = 2E\,(c \;\; s \;\; -is \;\; c)$
$\bar{u}_-(k')\gamma^\mu u_-(p') = 2E\,(c \;\; -s \;\; -is \;\; -c)$

Then

u d	u d	gives	
+ +	+ +	$(2E)^2 \cdot 2$	$= 2s$
+ −	+ −	$8E^2 \cdot c^2$	$= -2u$
− +	− +	$8E^2 \cdot c^2$	$= -2u$
− −	− −	$8E^2$	$= 2s$

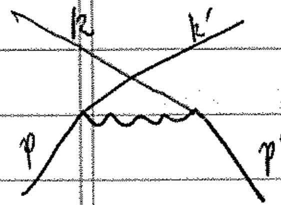
Then $iM(++--) = iM(--++) = ig^2 \frac{2s}{t} t^a_{ij} t^a_{kl}$
$iM(+--+) = iM(-++-) = -ig^2 \frac{2u}{t} t^a_{ij} t^a_{kl}$

On this page helicity of outgoing particles is the sign used
In $|M|^2$, we get $\mathrm{tr}(t^a t^b)\,\mathrm{tr}(t^a t^b) = (\frac{1}{2}\delta^{ab})^2 = \frac{8}{4}$
$a, b = 1, \dots, 8$

Then $\underbrace{\frac{1}{4}}_{\text{average over spin}} \cdot \underbrace{\frac{1}{9}}_{\text{average over color}} \sum_{\substack{\text{spin}\\ \text{color}}} |M|^2 = \frac{1}{36} \frac{2 \cdot 4 g^4 (s^2+u^2)}{t^2} \times 2 = \frac{4}{9} g^4 \frac{s^2+u^2}{t^2}$

Then $\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} \cdot \frac{4}{9} g^4 \frac{s^2+u^2}{t^2} = \frac{2\pi\alpha_s^2}{9s} \frac{s^2+u^2}{t^2}$

(d) For $uu \to uu$, additional diagram (u-channel)



$$= \frac{ig^2}{(p-k')^2}\, \bar{u}(k')\gamma^\mu u(p)\, \bar{u}(p')\gamma_\mu u(k)\, t^a_{il} t^a_{kj}$$

$$\bar{u}_+(k')\gamma^\mu u_+(p) = 2E\,(-s,\ c,\ ic,\ -s) \qquad \text{again } s = \sin\frac{\theta}{2}$$

$$\bar{u}_+(k)\gamma^\mu u_+(p') = 2E\,(s,\ c,\ -ic,\ -s) \qquad c = \cos\frac{\theta}{2}$$

$$\bar{u}_-(k')\gamma^\mu u_-(p) = 2E\,(-s,\ c,\ -ic,\ -s)$$

$$\bar{u}_-(k)\gamma^\mu u_-(p') = 2E\,(s,\ c,\ ic,\ -s)$$

Then including both t and u-channels

① $i\mathcal{M}(++--) = i\mathcal{M}(--++) = ig^2\left(\frac{2s}{t}\, t^a_{ij} t^a_{kl} - \frac{-2s}{u}\, t^a_{il} t^a_{kj}\right)$

② $i\mathcal{M}(+--+) = i\mathcal{M}(-++-) = -ig^2\, \frac{2u}{t}\, t^a_{ij} t^a_{kl}$

③ $i\mathcal{M}(-+-+) = i\mathcal{M}(+-+-) = -ig^2\, \frac{2t}{u}\, t^a_{il} t^a_{kj}$

(−) sign from fermion exchange

In these 3 lines, helicity of outgoing particles has the signs flipped

$$|①|^2 = 4g^4\left[\left(\frac{s^2}{t^2}+\frac{s^2}{u^2}\right)\underbrace{\left(\mathrm{tr}(t^a t^b)\right)^2}_{2} + 2\,\frac{s^2}{ut}\,\underbrace{\mathrm{tr}(t^a t^b t^a t^b)}_{-\frac{2}{3}}\right]$$

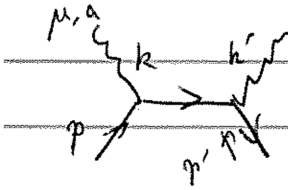
$$|②|^2 = 4g^4 \times \frac{u^2}{t^2} \times 2$$

$$|③|^2 = 4g^4 \times \frac{t^2}{u^2} \times 2$$

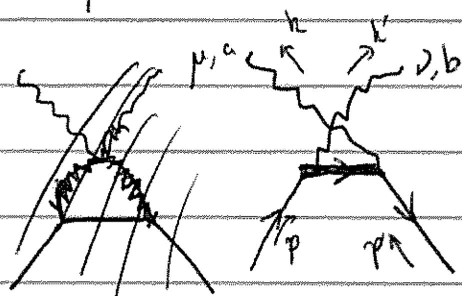
Then finally $\quad \frac{1}{4}\cdot\frac{1}{9}\sum|\mathcal{M}|^2 = \frac{4}{9}g^4\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - \frac{2}{3}\frac{s^2}{ut}\right)$

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha_s^2}{9s}\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - \frac{2}{3}\frac{s^2}{ut}\right)$$

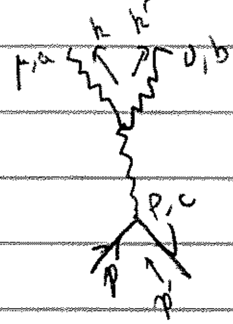
(e) Three diagrams



$$= \bar{v}(p')\, ig\gamma^\nu \varepsilon_\nu^*(k')\, t^b \frac{i(\not p - \not k)}{(p-k)^2}\, ig\gamma^\mu \varepsilon_\mu^*(k)\, t^a\, u(p)$$



$$= \bar{v}(p')\, ig\gamma^\mu \epsilon_\mu^*(k)\, t^a \frac{i(\not p - \not k')}{(p-k')^2}\, ig\gamma^\nu \epsilon_\nu^*(k')\, t^b\, u(p)$$



$$= g f^{abc} \epsilon^{*\mu}(k)\, \epsilon^{*\nu}(k') \left[g_{\mu\rho}(2k+k')_\nu + g_{\mu\nu}(k'-k)_\rho + g_{\nu\rho}(-2k'-k)_\mu \right]$$

$$\frac{-i}{(p+p')^2}\, \bar{v}(p')\, ig t^c \gamma^\rho u(p)$$

First consider the $(-+--)$ case.

t-channel:

$$iM_t = -ig^2\, t^b t^a\, \underset{\downarrow\, v_+(p')}{\bar{v}(p')}\, \underset{\downarrow\, \not\epsilon_+^*(k')}{\not\epsilon^*(k')}\, \frac{(\not p - \not k)}{(p-k)^2}\, \underset{\downarrow\, \not\epsilon_+^*(k)}{\not\epsilon^*(k)}\, \underset{\downarrow\, u_-(p)}{u(p)}$$

Note $\epsilon(k)$, $\epsilon(k')$, $p-k$ have no 0-th component.

We consider $\not a \not b \not c$ when a, b, c have no 0-th component

$$\not a \not b \not c = \begin{pmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{pmatrix}$$

where $\Sigma = a_i b_j c_k \sigma^i \sigma^j \sigma^k = a_i \sigma^i b_j c_k \left(\frac{1}{2}\{\sigma_j, \sigma_k\} + \frac{1}{2}[\sigma_j, \sigma_k] \right)$

$= a_i \sigma^i b_j c_k (\delta_{jk} + i\epsilon_{jkl}\sigma^l)$ ← repeat first line

$= a_i \sigma^i (b_j c_j) + i\, a_i b_j c_k \epsilon_{jkl}(\delta_{il} + i\epsilon_{ilt}\sigma^t)$

$= a_i \sigma^i (b_j c_j) + i\epsilon_{ijk} a_i b_j c_k - a_i b_j c_k \sigma^t(-\delta_{ji}\delta_{kt} + \delta_{jt}\delta_{ki})$

$$= (a\cdot\sigma)(b\cdot c) - (a\cdot c)(b\cdot\sigma) + (a\cdot b)(c\cdot\sigma) + i\epsilon_{ijk} a_i b_j c_k.$$

So $\not a \not b \not c = (b\cdot c)\not a - (a\cdot c)\not b + (a\cdot b)\not c + i\epsilon_{ijk} a_i b_j c_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Note $\bar v_+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_- = 0$ and $\bar v_+ \gamma^\mu u_- = 2E(0,-1,i,0) = -2E\,\epsilon_-(\hat z)$

Then $\bar v_+ \not\epsilon_+^*(k')(\not p - \not k)\not\epsilon_+^*(k) u_-$

$$= (p-k)\cdot\epsilon_+^*(k)\ \epsilon_+^*(k')\cdot(-2E\epsilon_-(\hat z)) - \epsilon_+^*(k')\cdot\epsilon_+^*(k)\ (p-k)\cdot(-2E\epsilon_-(\hat z))$$
$$+ (p-k)\cdot\epsilon_+^*(k')\ \epsilon_+^*(k)\cdot(-2E\epsilon_-(\hat z))$$
$$= -t\sin\theta$$

So $iM_t(-+--) = ig^2 t^b t^a \sin\theta$

Similar calculation gives $iM_u(-+--) = -ig^2 t^a t^b \sin\theta$

and $iM_s(-+--) = g^2 f^{abc} t^c \epsilon_+^*(k)\cdot\epsilon_+^*(k')\cdot\dfrac{-4E^2\sin\theta}{(2E)^2}$

$$= -g^2 f^{abc}\sin\theta$$

Since $[t^a, t^b] = if^{abc}t^c$, $\quad M_u + M_t + M_s\Big|_{(-+--)} = 0$

The case for $(-+++)$ is similar, with ϵ_-^*

(f) Now we only need to consider $(-+-+)$ and $(-++-)$.

Note in this case $\epsilon_+(k) = -\epsilon_-(k')$

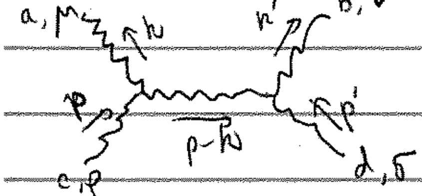
We get $iM_s(-+-+) = 0$, $iM_t(-+-+) + iM_u(-+-+) = ig^2\sin\theta\left(t^b t^a + t^a t^b \frac{t}{u}\right)$

$iM_s(-++-) = 0$, $iM_t(-++-) + iM_u(-++-) = -ig^2\sin\theta\left(t^b t^a \frac{u}{t} + t^a t^b\right)$

$$\frac{1}{9}\sum\left(|M(-+-+)|^2 + |M(-++-)|^2\right) = \frac{64g^4}{27}\left(\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\frac{t^2+u^2}{s^2}\right)$$

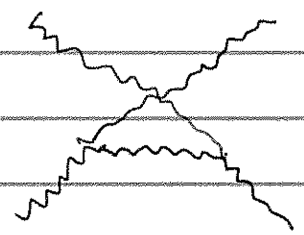
$$\frac{d\sigma}{d\cos\theta} = \frac{16\pi\alpha_s^2}{27s}\left(\frac{u}{t} + \frac{t}{u} - \frac{9}{4}\frac{t^2+u^2}{s^2}\right).$$

(g)(h)(i)(j)



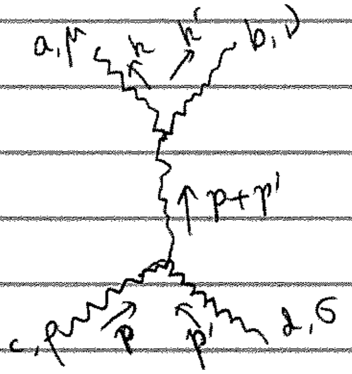
t-channel

$$= g f^{ace} \epsilon^*_\mu(k)\, \epsilon_\rho(p) \left[g^{\mu\lambda}(2k)^\rho + g^{\mu\rho}(-k-p)^\lambda + g^{\rho\lambda}(2p)^\mu \right] \frac{-i}{(p-k)^2}$$
$$\cdot g f^{dbe} \left[g^{\nu\sigma}(p'+k')^\lambda + g^{\nu\lambda}(-2k')^\sigma + g^{\lambda\sigma}(-2p')^\nu \right] \epsilon_\sigma(p')\, \epsilon^*_\nu(k')$$



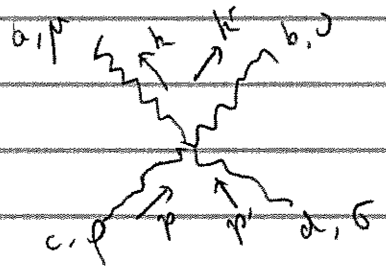
u-channel

= above-line w/ $a \leftrightarrow b$, $\mu \leftrightarrow \nu$, $k \leftrightarrow k'$



s-channel

$$= g^2 f^{cde} f^{bae} \epsilon_\rho(p)\, \epsilon_\sigma(p')\, \epsilon^*_\mu(k)\, \epsilon^*_\nu(k') \frac{-i}{(p+p')^2}$$
$$\cdot \left[g^{\rho\sigma}(p-p')^\lambda + g^{\sigma\lambda}(2p')^\rho + g^{\rho\lambda}(-2p)^\sigma \right]$$
$$\cdot \left[g^{\mu\nu}(k-k')^\lambda + g^{\mu\lambda}(-2k)^\nu + g^{\lambda\nu}(2k')^\mu \right]$$



$$= \epsilon^*_\mu(k)\, \epsilon_\rho(p) \left[-ig^2 f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) - ig^2 f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \right.$$
$$\left. - ig^2 f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right] \epsilon_\sigma(p')\, \epsilon^*_\nu(k')$$

When summing up those four amplitudes, it is helpful to organize into common $f^{\alpha\beta\gamma} f^{\delta\lambda\gamma}$ terms. So we divide the 4-pt amplitude into 3 pieces, and add each into corresponding s, t, u amplitudes

Let's call the results M_s, M_t, M_u, so $M = M_s + M_t + M_u$

Then now, we can just compute separately

1) vector-index computations (all ϵ_μ's, $g^{\mu\nu}$, p^μ etc)

2) color-index computations (f^{abc}'s)

$p = E(1,0,0,1)$, $p' = E(1,0,0,-1)$

$k = E(1,s,0,c)$, $k' = E(1,-s,0,-c)$. $c = \cos\theta$, $s = \sin\theta$

$\epsilon_+(p) = \frac{1}{\sqrt{2}}(0,1,i,0)$ $\quad\quad$ $\epsilon_-(p) = \frac{1}{\sqrt{2}}(0,1,-i,0)$

$\epsilon_+(p') = \frac{1}{\sqrt{2}}(0,-1,i,0)$ $\quad\quad$ $\epsilon_-(p') = \frac{1}{\sqrt{2}}(0,-1,-i,0)$

$\epsilon_+^*(k) = \frac{1}{\sqrt{2}}(0,c,-i,-s)$ $\quad\quad$ $\epsilon_-^*(k) = \frac{1}{\sqrt{2}}(0,c,i,-s)$

$\epsilon_+^*(k') = \frac{1}{\sqrt{2}}(0,-c,-i,s)$ $\quad\quad$ $\epsilon_-^*(k') = \frac{1}{\sqrt{2}}(0,-c,i,s)$.

Also note all ϵ's and $p-k$, $p-k'$ do not have 0-th component

(g) (h) What remains is computations!

For $(++++)$ and $(+++-)$, you can check $M_s = M_t = M_u = 0$ individually.

(i) For $(++--)$,
$$iM_s = -2ig^2 f^{cde} f^{abe} \cos\theta$$
$$iM_t = 2ig^2 f^{ace} f^{bde} \frac{(2-\cos\theta)(1+\cos\theta)}{1-\cos\theta}$$
$$iM_u = 2ig^2 f^{bce} f^{ade} \frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

Use Jacobi's identity $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$ to turn M_s into two ff's like in M_t, M_u

Then one gets $iM = -2ig^2 \left(f^{ace} f^{bde} \frac{s}{t} + f^{bce} f^{ade} \frac{s}{u} \right)$

(j) We see above non-zero amplitudes are from 2 +'s and 2 -'s.
We have $(++--)$, $(--++)$, $(+-+-)$, $(+--+)$, $(-+-+)$, $(-++-)$. They are related by crossing-symmetry.

$$iM(cd\to ab;++--;s,t,u) = -2ig^2\left(f^{ace}f^{bde}\frac{s}{t} + f^{bce}f^{ade}\frac{s}{u}\right)$$

$$iM(cd\to ab;--++;s,t,u) = iM(ab\to cd;++--;s,t,u)$$
$$= -2ig^2\left(f^{ace}f^{bde}\frac{s}{t} + f^{bce}f^{ade}\frac{s}{u}\right).$$

$$iM(cd\to ab;+--+;s,t,u) = iM(cb\to ad;++--;u,t,s)$$
$$= 2ig^2\left(f^{ace}f^{bde}\frac{u}{t} + f^{abe}f^{cde}\frac{u}{s}\right).$$

$$iM(cd\to ab;-++-;s,t,u) = iM(ad\to cb,++--;u,t,s)$$
$$= 2ig^2\left(f^{ace}f^{bde}\frac{u}{t} + f^{ace}f^{bde}\frac{u}{s}\right).$$

$$iM(cd\to ab;+-+-;s,t,u) = iM(ca\to db;++--;t,s,u)$$
$$= -2ig^2\left(f^{cde}f^{abe}\frac{t}{s} - f^{ade}f^{bce}\frac{t}{u}\right)$$

$$iM(cd\to ab;-+-+;s,t,u) = iM(bd\to ac;++--;t,s,u)$$
$$= -2ig^2\left(f^{abe}f^{cde}\frac{t}{s} - f^{ade}f^{bce}\frac{t}{u}\right)$$

Square the amplitudes and sum over initial and final colors.
Note that
$$f^{abe}f^{cde} = (t_G^b t_G^d)_{ac} = (t_G^a t_G^c)_{bd} = -(t_G^a t_G^d)_{bc} = -(t_G^b t_G^c$$

$$\frac{1}{4}\frac{1}{d(G)^2}\sum_{\substack{\text{spins}\\\text{colors}}}|M|^2 = \frac{2g^4}{d(G)^2}\left[72\left(\frac{s^2}{t^2}+\frac{s^2}{u^2}+\frac{u^2}{t^2}+\frac{u^2}{s^2}+\frac{t^2}{s^2}+\frac{t^2}{u^2}\right)+36\left(\frac{2s^2}{ut}+\frac{2u^2}{st}+\frac{2t^2}{su}\right)\right]$$
$$= \frac{9g^4}{2}\left(3-\frac{st}{u^2}-\frac{ut}{s^2}-\frac{us}{t^2}\right)$$
(We use $s+t+u=0$ for massless gluons)

Finally, $$\frac{d\sigma}{d\cos\theta} = \frac{9\pi\alpha_s^2}{4s}\left(3-\frac{st}{u^2}-\frac{ut}{s^2}-\frac{us}{t^2}\right).$$