Problem Set #5 - Solutions

1. (a) From Peskin (7.77), $\alpha(q^2) = \dfrac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]}$

Where from Peskin (7.91), $\hat{\Pi}_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \log\left(\frac{m^2}{m^2-x(1-x)q^2}\right)$

Note for $m^2 \gg |q^2|$, $\hat{\Pi}_2(q^2) = 0 \Rightarrow$ no effect.

for $m^2 \ll -q^2$, $\hat{\Pi}_2 = \frac{\alpha}{3\pi}\left(\log\frac{-q^2}{m^2} - \frac{5}{3}\right)$

Note $\alpha^{-1}(q^2) = \alpha^{-1}\left(1 - \hat{\Pi}_2(q^2)\right)$

For electron only, $\alpha^{-1}(Q) = \alpha_0^{-1}\left(1 - \frac{\alpha}{3\pi}\log\frac{Q^2}{m^2} - \frac{5}{3}\right)$

(b) Add muons, $\alpha^{-1}(Q) = \alpha_0^{-1}\left(1 - \hat{\Pi}_e - \hat{\Pi}_\mu\right)$

$\hat{\Pi}_e = \hat{\Pi}_2(q^2, m_e^2)$, $\hat{\Pi}_\mu = \hat{\Pi}_2(q^2, m_\mu^2)$

(c), u, c: charge $+\frac{2}{3}$

d, s, b: charge $-\frac{1}{3}$.

so $\alpha^{-1}(Q) = \alpha_0^{-1}\left(1 - \hat{\Pi}_e - \hat{\Pi}_\mu - \hat{\Pi}_\tau - \sum_{\substack{u,d\\ s,c,b}} \hat{\Pi}_q\right)$

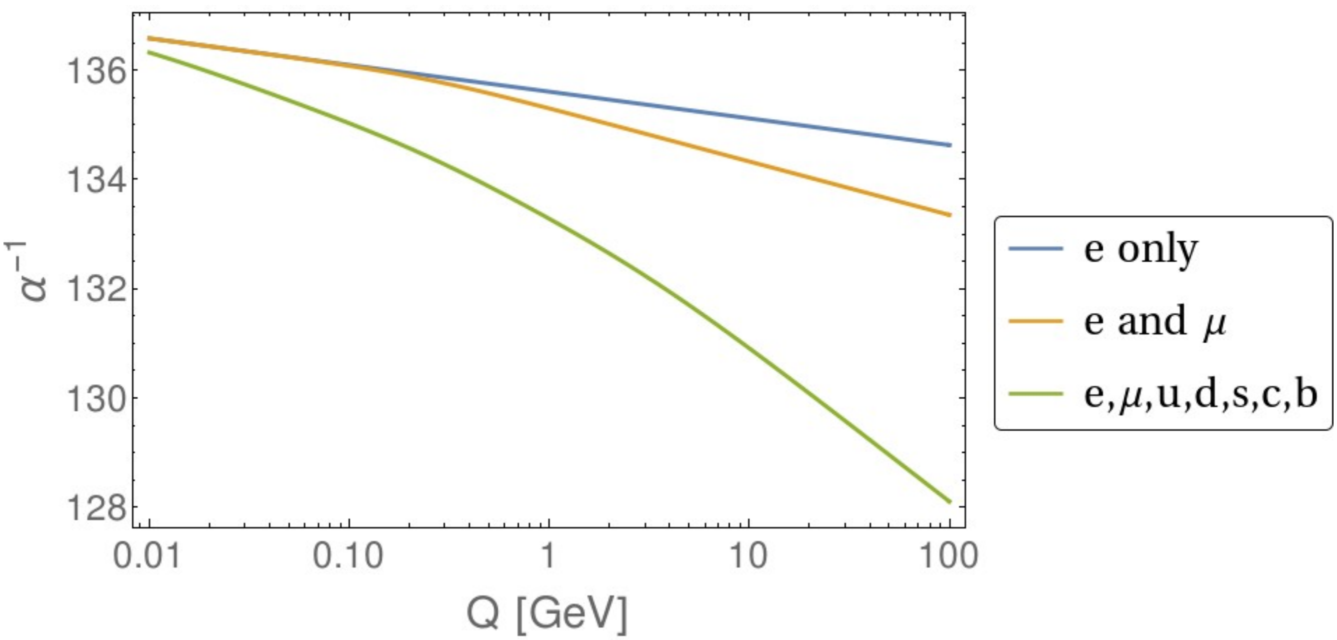
$\hat{\Pi}_u = 1.04 \times 3 \times \left(\frac{2}{3}\right)^2 \hat{\Pi}_2(q^2, m_u^2)$

$\hat{\Pi}_d = 1.04 \times 3 \times \left(\frac{1}{3}\right)^2 \hat{\Pi}_2(q^2, m_d^2)$

and so on.

(d) $\alpha^{-1}(m_Z) = 128.2$

(a),(b),(c)



136
134
132
130
128
α^{-1}
0.01
0.10
1
10
100
Q [GeV]
e only
e and μ
e,μ,u,d,s,c,b

2. (a) $\sum_{n=0}^{\infty} P(n) = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1.$

$$\langle n \rangle = \sum_{n=0}^{\infty} n P(n) = \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = \lambda.$$

(b) Note $n \lambda^n e^{-\lambda} = \left(\lambda \frac{d}{d\lambda} + \lambda\right)\left(\lambda^n e^{-\lambda}\right)$

Then $\langle n^2 \rangle = \sum_{n=0}^{\infty} \frac{n^2 \lambda^n}{n!} e^{-\lambda} = \left(\lambda \frac{d}{d\lambda} + \lambda\right) \sum_{n=0}^{\infty} \frac{n \lambda^n e^{-\lambda}}{n!}$

$$= \left(\lambda \frac{d}{d\lambda} + \lambda\right) \langle n \rangle = \lambda + \lambda^2$$

$$\langle n^3 \rangle = \left(\lambda \frac{d}{d\lambda} + \lambda\right) \langle n^2 \rangle = \lambda + 3\lambda^2 + \lambda^3$$

So $\langle (n-\lambda)^2 \rangle = \langle n^2 \rangle - 2\lambda \langle n \rangle + \lambda^2 = \lambda$

$$\langle (n-\lambda)^3 \rangle = \langle n^3 \rangle - 3\lambda \langle n^2 \rangle + 3\lambda^2 \langle n \rangle - \lambda^3 = \lambda$$

For a Gaussian distribution with μ, σ^2

we have $\langle x \rangle = \mu$

$\langle x^2 \rangle = \mu^2 + \sigma^2$ $\qquad \langle (x-\mu)^2 \rangle = \sigma^2$

$\langle x^3 \rangle = \mu^3 + 3\mu\sigma^2$ $\qquad \langle (x-\mu)^3 \rangle = 0$

In the large λ limit, $\mu = \sigma^2 = \lambda$. $\quad \langle x \rangle \to \lambda$ $\quad$ and $\quad \langle n \rangle = \lambda$

$\langle x^2 \rangle \to \lambda^2$ $\qquad \langle n^2 \rangle \to \lambda^2$

$\langle x^3 \rangle \to \lambda^3$ $\qquad \langle n^3 \rangle \to \lambda^3$

(c) See p. 32 of Peskin.

The solution of the inhomogeneous Klein-Gordon equation is given by the retarded Green's function.

$$\phi(x) = \phi_0(x) + i \int d^4y \, D_R(x-y) \, j(y)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{\dagger} e^{ip\cdot x}\right) + i \int d^4y \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} j(y)$$

(d) $j(q) = \int d^4y \, e^{iq\cdot y} j(y)$.

Since $j(x) = 0$ for $t > t_2$, $\int_{-\infty}^{\infty} dt \, e^{iq_0 t} j(\vec{q}, t) = \int_{-\infty}^{t_2} e^{iq_0 t} j(\vec{q}, t)$

for $\mathrm{Im}\, q^0 < 0$, as $\mathrm{Im}\, q^0 \to -\infty$, $|e^{iq_0 t}| = e^{-(\mathrm{Im}\, q_0) t} < e^{-(\mathrm{Im}\, q_0) t_2}$ for $t <$

So $j(q)$ has a divergence as worst as $e^{iq_0 t_2}$, as $\mathrm{Im}\, q^0 \to -\infty$.

This implies

$$(\partial_x^2 + m^2)\left[i \int d^4y \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} j(y) \right]$$

$$= i \int d^4y \int \frac{d^4p}{(2\pi)^4} (-p^2 + m^2) \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} j(y)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} j(p)$$

In the p^0 integral, by closing the contour below, the divergence in $j(p)$ in the lower half-plane is more than contained by $e^{-ip\cdot x}$, if $x^0 > t_2$, so thi this evaluates to zero.

Using the representation of the retarded Green's function as in Peskin (2.54), for $x^0 > t_2$,

$$\phi(x) = \phi_0(x) + i \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot x} j(p) - e^{ip\cdot x} j(p)^* \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left\{ \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} j(p) \right) e^{-ip\cdot x} + h.c. \right\}$$

This means that the vacuum Hamiltonian $H_0 = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p$ is now changed into

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a_{\vec{p}}^\dagger - \frac{i}{\sqrt{2E_{\vec{p}}}} j^*(p) \right) \left(a_p + \frac{i}{\sqrt{2E_{\vec{p}}}} j(p) \right)$$

Then,

$$E = \langle 0 | H | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |j(p)|^2$$

(e) From the H above, the "number" operator becomes

$$N = \int \frac{d^3p}{(2\pi)^3} \left(a_p^\dagger - \frac{i}{\sqrt{2E_p}} j^*(p) \right) \left(a_p + \frac{i}{\sqrt{2E_p}} j(p) \right)$$

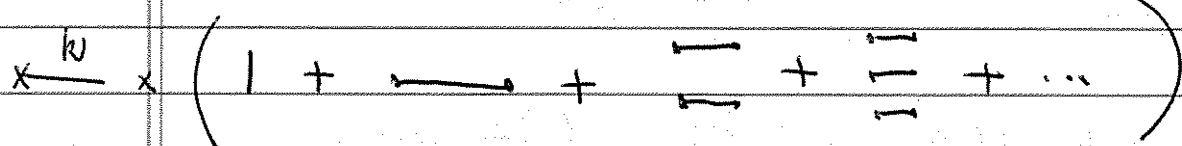
so $\langle N \rangle = \langle 0 | N | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_p} |j(p)|^2 .$

(f) $\mathcal{L} = \mathcal{L}_0 + j(x)\phi(x) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + j(x)\phi(x)$

$j \times \xleftarrow{\ p\ } \; : \; i j(-p)$ $\qquad i \int d^4x \, j(x)\phi(x) = i \int \frac{d^4p}{(2\pi)^4} j(p)\phi(-p)$

$$\times\!\!-\!\!\!-\!\!\times = \cancel{i^2 |j(p)|^2} \; i^2 \int \frac{d^4k}{(2\pi)^4} \delta^{(4)}(k^2 - m^2) |j(k)|^2 = i^2 \underbrace{\boxed{\int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E_k} |j(k)|^2}}_{\langle N \rangle \equiv \lambda}$$

The amplitude for creating a single particle at momentum k is



$= i\mathcal{M}_k$

The n-th order in this series has an overall constant

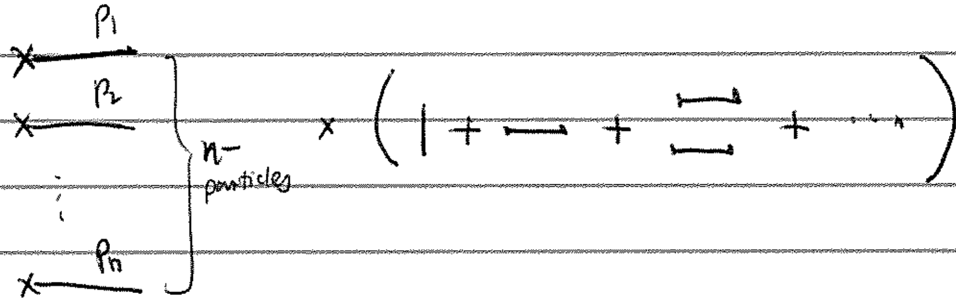
$$\underbrace{i^{2n}}_{(-1)^n} \times \underbrace{\frac{1}{(2n)!}}_{\text{From Taylor series}} \times \underbrace{(2n-1)(2n-3)\cdots(3)(1)}_{\#\text{ of contractions}} = \frac{(-1)^n}{2^n n!}$$

Another way to derive this is to count the symmetry factor of the diagram (see Peskin p93).

$$S = \underbrace{n!}_{\text{the } n\text{-pairs can be interchanged.}} \times \underset{\substack{\downarrow \\ \text{for each pair, the two vertices can be interchanged}}}{2^n}$$

Thus $|\mathcal{M}|^2 = \lambda \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \lambda^n \right)^2 = \lambda \left(\sum_{n=0}^{\infty} \frac{(-\lambda/2)^n}{n!} \right)^2 = \lambda e^{-\lambda}$

(g)

$i\mathcal{M} =$ 

$$|\mathcal{M}|^2 = \frac{\lambda^n}{n!} \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \lambda^m \right)^2 = \frac{\lambda^n}{n!} e^{-\lambda}$$

exchanging the n-vertices

(h) From above, we see

$$\langle n \rangle = \sum n P_n = \lambda = \int \frac{d^3p}{(2\pi)^3} \frac{|j(p)|^2}{2E_p}$$

Then expected total energy production is $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |j(p)|^2$, as in (d).

3. (a). Contract $\mathrm{tr}(t_r^a t_r^b) = C(r)\,\delta_{ab}$ w/ δ^{ab}

$$C(r)\,\delta^{aa} = \mathrm{tr}\,(t_r^a t_r^a)$$

$$C(r)\,d_G = C_2(r)\,\mathrm{tr}\,\mathbb{1} = C_2(r)\,d_r$$

(b) There can be different answers.
Standard choices are

$SU(2)$: $\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$SU(3)$: See p502 of Peskin ("Gell-Mann Matrices").

(c) $C_2(N) = \dfrac{d_G}{d_N}\,C(N) = \dfrac{N^2-1}{N} \times \dfrac{1}{2} = \dfrac{N^2-1}{2N}$

$\left(d_G = N^2-1,\ d_N = N,\ C(N) = \frac{1}{2}\right)$

(d) r of G acting on ξ_α : $\xi_\alpha \to (1 + i\alpha^a t_r^a)\,\xi_\alpha$
Complex conjugation is

$$\xi_a^\dagger \to \left(1 - i\alpha^a (t_r^a)^*\right)\xi_a^\dagger = (1 + i\alpha^a t_{\bar{r}}^a)\,\xi_a^\dagger$$

Thus $t_{\bar{r}}^a = -(t_r^a)^* = -(t_r^a)^T$ since $(t_r^a)^\dagger = t_r^a$

(e) $\Sigma_{\alpha\beta} \to (1 + i\,t^a_{r_1\otimes r_2}\,\alpha^a)\,\Sigma_{\alpha\beta}$

$$= (1 + i\alpha^a t_{r_1}^a + i\alpha^a t_{r_2}^a)\,\Sigma_{\alpha\beta} = \left(1 + i\alpha^a\left(t_{r_1}^a \otimes \mathbb{1}_{d_2} + \mathbb{1}_{d_1} \otimes t_{r_2}^a\right)\right)\Sigma_{\alpha\beta}$$

so $t^a_{r_1\otimes r_2} = t_{r_1}^a \otimes \mathbb{1}_{d_2} + \mathbb{1}_{d_1} \otimes t_{r_2}^a$

$$\begin{aligned}\mathrm{tr}\,(t^a_{r_1\otimes r_2}\,t^a_{r_1\otimes r_2}) &= \mathrm{tr}\left[(t_{r_1}^a t_{r_1}^a)\otimes \mathbb{1}_{d_2} + 2\,t_{r_1}^a \otimes t_{r_2}^a + \mathbb{1}_{d_1} \otimes (t_{r_2}^a t_{r_2}^a)\right] \\ &= C_2(r_1)\,d_{r_1} d_{r_2} + 0 + d_{r_1}\,C_2(r_2)\,d_{r_2} \\ &= (C_2(r_1) + C_2(r_2))\,d_1 d_2\end{aligned}$$

(f) $\mathrm{tr}(t_R^a t_R^a) = \sum_j \mathrm{tr}\left(t_{r_j}^a t_{r_j}^a \right) = \sum_j C_2(j) d_j$

By (e), this is equal to $\left(C_2(r_1) + C_2(r_2)\right) d_1 d_2$.

(g) $N \otimes \bar{N} = 1 \oplus N^2-1$

Use (f)

$$\left(\frac{N^2-1}{2N} + \frac{N^2-1}{2N}\right) N \cdot N = 0 + C_2(G)\cdot(N^2-1) \quad \Rightarrow \quad C_2(G) = N$$

Note $C_2 = 0$ for the trivial representation.

(h) One aligns the $SU(2)$ subgroup of $SU(2)$ as the first two components of N representation. Under this $SU(2)$ all other components transform trivially.

So $N = 2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{(N-2)}$ under $SU(2)$.

When t^3 is the representation matrix of J^3 in the fundametal representation of the $SU(2)$ subgroup of $SU(N)$

$$t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{tr}(t^3 t^3) = \frac{1}{2} = C(N)$$

Now, note for $SU(N)$, $N = 2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{(N-2)}$ under $SU(2)$

$$\bar{N} = \bar{2} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{(N-2)}$$

note $2 = \bar{2}$, $2 \otimes 2 = 3 \oplus 1$

Then $N \otimes \bar{N} = 3 \oplus \underbrace{2 \oplus \cdots \oplus 2}_{2(N-2)} \oplus 1 \oplus \cdots \oplus 1$

$= G \oplus 1$

Note $C(r) = 0$ for r singlet (the "1")

Since for 3 rep of $SU(2)$, $C(r) = 2$

2 rep of $SU(2)$, $C(r) = \frac{1}{2}$

So $C(G) = 2 + 2(N-2) \times \frac{1}{2} = N$

(i) S, A are symmetric and anti-symmetric 2-index tensors of $SU(N)$, and they can themselves be viewed as symmetric and anti-symmetric $N*N$ matrices.

$$\#\text{ of Symmetric matrices} = \underbrace{N}_{\text{diagonal}} + \underbrace{\frac{N^2-N}{2}}_{\text{upper triangular}} = \frac{N(N+1)}{2} = \dim(S)$$

$$\#\text{ of Anti-symmetric matrices} = \underbrace{\frac{N^2-N}{2}}_{\text{upper triangular}} = \frac{N(N-1)}{2} = \dim(A)$$

Then $(N\otimes N)_{sym} = \Big[(2\oplus \underbrace{1\oplus\cdots\oplus 1}_{N-2}) \otimes (2\oplus \underbrace{1\oplus\cdots\oplus 1}_{N-2})\Big]_{sym}$

$$= (2\otimes 2)_{sym} \oplus \big[2\otimes(1\oplus\cdots\oplus 1) + (1\oplus\cdots\oplus 1)\otimes 2\big]_{sym}$$

$$\oplus \Big[(\underbrace{1\oplus\cdots\oplus 1}_{N-2})\otimes(\underbrace{1\oplus\cdots\oplus 1}_{N-2})\Big]_{sym}$$

$$= 3 \oplus (\underbrace{2\oplus\cdots\oplus 2}_{N-2}) \oplus (\underbrace{1\oplus\cdots\oplus 1}_{\frac{(N-2)(N-1)}{2}})$$

So $C(S) = 2 + (N-2)\frac{1}{2} = \frac{N+2}{2}$

Similarly, $(N\otimes N)_{anti} = 1 \oplus (\underbrace{2\oplus\cdots\oplus 2}_{N-2}) + (\underbrace{1\oplus\cdots\oplus 1}_{\frac{(N-2)(N-3)}{2}})$

$C(A) = \frac{N-2}{2}$

Then from $C(r)\, d_G = C_2(r)\, d_r$

$$C_2(S) = \frac{N+2}{2} \cdot \frac{N^2-1}{N(N+1)/2} = \frac{(N+2)(N-1)}{N}$$

$$C_2(A) = \frac{N-2}{2} \cdot \frac{N^2-1}{N(N-1)/2} = \frac{(N+2)(N+1)}{N}$$

(j) Consider $t^a_{r_1\otimes r_2}\, t^a_{r_1\otimes r_2}$, (recall $t^a_{r_1\otimes r_2} = t_{r_1}{}^a \otimes 1_{d_2} + 1_{d_1} \otimes t_{r_2}{}^a$)

$$= t_{r_1}{}^a t_{r_1}{}^a \otimes 1_{d_2} + 1_{d_1} \otimes t_{r_2}{}^a t_{r_2}{}^a + 2\, t_{r_1}{}^a \otimes t_{r_2}{}^a$$

$$= \left[C_2(r_1) + C_2(r_2)\right] 1_{d_1 \times d_2} + 2\, t_{r_1}{}^a \otimes t_{r_2}{}^a$$

Also $r_1 \otimes r_2 = \sum_j r_j$ so $t^a_{r_1\otimes r_2}\, t^a_{r_1\otimes r_2} = \sum_j t_{r_j}{}^a t_{r_j}{}^a = \sum_j C_2(r_j)\, 1_j$, where 1_j is a diagonal matrix of size $d_1 d_2 \times d_1 d_2$, with 1's at the j-th block-diagonal position, and 0 everywhere else.

Thus $t_{r_1}{}^a \otimes t_{r_2}{}^a$ has eigenvalues $\frac{1}{2}\left(C_2(r_j) - C_2(r_1) - C_2(r_2)\right)$.

At least one of these eigenvalues is negative, since $\mathrm{tr}\left(t_{r_1}{}^a \otimes t_{r_2}{}^a\right) = 0$