$\epsilon = 2 - \frac{d}{2}$.

1. (a) $$\Sigma_2(p) = \overset{e^2}{(4\pi\alpha)} \cdot \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{[(1-x)m^2 + x\mu^2 - x(1-x)p^2]^{2-d/2}} \left[(4-2\epsilon)m - (2-2\epsilon)x\not p\right]$$

Where μ is photon mass for infrared regularization. (See part (e)).

Peskin (A.52) $$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + O(\epsilon)\right)$$

So it is important to keep the $O(\epsilon)$ terms in $[(4-2\epsilon)m - (2-2\epsilon)x\not p]$ because they can give $O(1)$ terms upon multiplying $\frac{1}{\epsilon}$.

$$-i\Sigma = \longleftarrow = -i\Sigma_2 + \longleftarrow\!\otimes\!\longleftarrow = -i\Sigma_2 + i(\not p\,\delta_2 - \delta_m).$$

Renormalization conditions

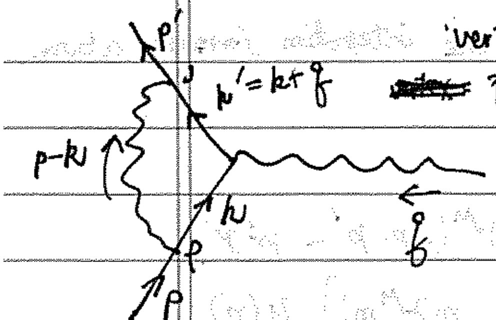
$$\frac{d}{d\not p}\Sigma(\not p)_{\not p = m} = 0 \Rightarrow \delta_2 = \frac{d}{d\not p}\Sigma_2(m)$$

$$= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{((1-x)^2m^2+x\mu^2)^{2-d/2}} \times \left[(2-2\epsilon)x - \epsilon\frac{2x(1-x)m^2}{(1-x)^2m^2+x\mu^2} \times (4-2x-2\epsilon(1-x))\right]$$

$$\Sigma(\not p = m) = 0 \Rightarrow m\delta_2 - \delta_m = \Sigma_2(m)$$

$$\delta_m = m\delta_2 - \Sigma_2(m) = -\frac{e^2 m}{(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{((1-x)^2m^2+x\mu^2)^{2-d/2}}\left[(4-2\epsilon) - \epsilon\frac{2x(1-x)m^2}{(1-x)^2m^2+x\mu^2}(4-2x-2\epsilon(1-x))\right]$$

(b) This is same as section 6.3 of Peskin, except use dimensional regularization.



vertex $= \bar u(p')(-ie\,\delta\Gamma^\mu(p',p))u(p)$ (ignoring μ^2 for now)

$$\bar u(p')\,\delta\Gamma^\mu(p',p)\,u(p)$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar u(p')(-ie\gamma^\nu)\frac{i(\not k'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not k+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p)$$

Use Peskin (5.9). $\gamma^\nu \not k'\gamma^\mu \not k\gamma_\nu = -2\not k\gamma^\mu\not k'$. $\gamma^\nu\not k'\gamma^\mu\gamma_\nu = 4k'^\mu$

$$\therefore\ \bar u(p')\,\delta\Gamma^\mu u(p) = 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar u(p')[\not k\gamma^\mu\not k' + m^2\gamma^\mu - 2m(k+k')^\mu]u(p)}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

$$= 4ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4k}{(2\pi)^4}\frac{\bar u(p')[\not k\gamma^\mu\not k'+m^2\gamma^\mu-2m(k+k')^\mu]u(p)}{x(k^2-m^2)+y(k'^2-m^2)+z(k-p)^2+i\epsilon}$$

denominator $D = k^2 + 2k(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\epsilon$

Let $l = k + yq - zp$. $D = l^2 - \Delta + i\epsilon$

with $\Delta = (yq - zp)^2 + (x+y)m^2 - yq^2 - zp^2$

$= (y^2 - y)q^2 + (z^2 - z)p^2 - 2yz\, p\cdot q + \overbrace{(x+y)}^{1-z}m^2$

use $m^2 = p'^2 = (p+q)^2 = m^2 + 2p\cdot q + q^2$, $p^2 = m^2 \Rightarrow 2p\cdot q = -q^2$

$$\Delta = (y^2 - y)q^2 + (z^2 - z + 1 - z)m^2 + yz q^2$$
$$= -xy\, q^2 + (1-z)^2 m^2$$

Using $\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu}{D^3} = 0$. $\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu l^\nu}{D^3} = \frac{1}{4} \int \frac{d^4 l}{(2\pi)^4} \frac{g^{\mu\nu} l^2}{D^3}$

Numerator $= \bar{u}(p')\left[\not{k} \gamma^\mu \not{k}' + m^2 \gamma^\mu - 2m(k+k')^\mu \right] u(p)$

$\rightarrow \bar{u}(p')\Big[-\frac{1}{2}\gamma^\mu l^2 + (-y\not{q} + z\not{p})\gamma^\mu\left((1-y)\not{q} + z\not{p}\right)$
$+ m^2\gamma^\mu - 2m\left((1-2y)q^\mu + 2z p^\mu\right)\Big] u(p)$

Performing integration:

use (A.44) $\int \frac{d^d l}{(2\pi)^d} \cdot \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}}$

(A.45) $\int \frac{d^d l}{(2\pi)^d} \cdot \frac{l^2}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1}$

(c) We perform simplification of numerator before chugging the integration formulas above.

$\not{p}\, u(p) = m\, u(p)$

$\bar{u}(p')\not{p}' = \bar{u}(p') m$

Note $\not{p}\gamma^\mu \not{p}' = (2p^\mu - \gamma^\mu \not{p})\not{p}' = 2\not{p}' p^\mu - \gamma^\mu(2p\cdot p' - \not{p}'\not{p})$

$\bar{u}(p')\not{p}\gamma^\mu \not{p}' u(p) = \bar{u}(p')\left[2m p^\mu - 2(p\cdot p')\gamma^\mu + (2p'^\mu m - m\gamma^\mu m)\right] u(p)$

$$= \bar{u}(p')\left[2m(p+p')^\mu - (m^2 + 2(p\cdot p'))\gamma^\mu \right] u(p)$$

$\bar{u}(p')\gamma^\mu \not{p}' u(p) = \bar{u}(p')(2p'^\mu - m\gamma^\mu) u(p)$

$\bar{u}(p')\not{p}\gamma^\mu u(p) = \bar{u}(p')(2p^\mu - m\gamma^\mu) u(p)$

After simplification w/ $x+y+z=1$.

$$\text{numerator} = \bar{u}(p')\left[\gamma^\mu\left(-\tfrac{1}{2}\ell^2+(1-x)(1-y)q^2+(1-2z-z^2)m^2\right)+mz(z-1)(p+p')^\mu + q^\mu m(x-y)(z-2)\right]u(p).$$

Denominator invariant upon exchanging $x \Leftrightarrow y$ $\Rightarrow$ q^μ term vanishes upon $\int dx\,dy$.

Use Gordan identity (6.32) $\quad \bar{u}(p')\frac{(p'+p)^\mu}{2m}u(p) = \bar{u}(p')\gamma^\mu u(p) - \bar{u}(p')\frac{i\sigma^{\mu\nu}q_\nu}{2m}u(p)$

we get

$$\bar{u}(p')\delta\Gamma u(p) = 4ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\cdot\frac{1}{D^3}\,\bar{u}(p')\Big[\gamma^\mu\left(-\tfrac{1}{2}\ell^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2\right) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}\left(2m^2z(1-z)\right)\Big]u(p)$$

(c) $$\bar{u}(p')\delta\Gamma u(p) = 4ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\bar{u}(p')\left[A\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}B\right]u(p)$$

$$A = -\frac{1}{2}\frac{i}{(4\pi)^{d/2}}\overset{(2-\epsilon)}{\left(\frac{d}{2}\right)}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} + \left[(1-x)(1-y)q^2+(1-4z+z^2)m^2\right]\frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)$$

$$B = 2m^2z(1-z)\frac{-i}{(4\pi)^2}\frac{\Gamma(1)}{\Gamma(3)}\cdot\frac{1}{\Delta}$$

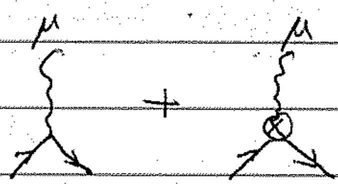
$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\left(m^2(1-z)^2-q^2xy\right)^{2-d/2}} + 2\,\frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy}\right]$$

$$F_2(q^2) = \frac{2e^2}{(4\pi)^{d/2}}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2-q^2xy}.$$

[This has no $\frac{1}{\epsilon}$ term so F_2 is finite because this is the electron's magnetic moment.]

Putting in photon mass μ for IR divergence regulerization, $\Delta \to \Delta + \mu^2 z$

Add δ_1 contribution:



$$= -ie\gamma^\mu - ie\gamma^\mu\delta_1 = -ie\Gamma^\mu$$

Renormalization condition: $\Gamma^\mu(q=0) = 1$.

so we have

$$\delta_1 = -\delta F_1(q^2=0) = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\left[(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\left(m^2(1-z)^2+\mu^2z\right)^{2-d/2}} + \frac{2m^2(1-4z+z^2)}{m^2(1-z)^2+\mu^2z}\right]$$

(d) The $\frac{1}{\epsilon}$ term in $\delta_1 = -e^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\cdot\frac{2}{\epsilon}$

$$= -\frac{2e^2}{\epsilon}\int_0^1 dz\int_0^{1-z} dy\,1 = -\frac{2e^2}{\epsilon}\int_0^1 dz(1-z) = -\frac{e^2}{\epsilon}$$

The $\frac{1}{\epsilon}$ term in $\delta_2 = -e^2\int_0^1 dx\,\frac{1}{\epsilon}\times 2x = -\frac{e^2}{\epsilon}$

Both also have a factor of $\frac{1}{(4\pi)^2}$ in front.

(e) At $q^2=0$, without μ (photon mass)

the second term of $\delta_1 = -\frac{2e^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{-2+(1-z)(3-z)}{(1-z)^2}$

$$= -\frac{2e^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\,\frac{-2+(1-z)(3-z)}{(1-z)^2}$$

$$= \frac{-2e^2}{(4\pi)^2}\underbrace{\int_0^1 dz\,\frac{-2}{(1-z)}}_{\text{divergent.}} + \text{finite terms}$$

Similarly, at $q^2=0$, $\mu=0$, δ_2 contains a term proportional to $\int_0^1 dx\,\frac{x}{m^2(1-x)} = \infty$

(f)

$$\int_0^1 dz\,2(2z-1)\log\left(m^2(1-z)^2+\mu^2 z\right) = \int_0^1 d\left(2z(z-1)\right)\log\left(m^2(1-z)^2+\mu^2 z\right)$$

$$= 2z(z-1)\Big|_0^1 - \int_0^1 2z(z-1)\cdot d\left(\log(m^2(1-z)^2+\mu^2 z)\right)$$

$$= -\int_0^1 2z(z-1)\,\frac{-2(1-z)m^2+\mu^2}{m^2(1-z)^2+\mu^2 z}$$

$$= 2\int_0^1 dz\left[(1-z) - \frac{(1-z)(1-z^2)m^2}{m^2(1-z)^2+\mu^2 z}\right]$$

$$\frac{2m^2(1-4z+z^2)(1-z)}{m^2(1-z)^2+\mu^2 z} + \frac{2z(1-z)m^2(4-2z)}{m^2(1-z)^2+\mu^2 z} = \frac{2(1-z)(1-z^2)m^2}{m^2(1-z)^2+\mu^2 z}$$

2. (a) From problem 1, $\frac{-i g_{\nu\rho}}{(k-p)^2} \longrightarrow \frac{-i}{(k-p)^2 - M^2}\left(g_{\nu\rho} - \frac{(k-p)_\mu (k-p)_\nu}{M^2}\right)$

Then $\Delta = (1-z)^2 m^2 + z M^2 - xy q^2$

We also have an extra term from $\frac{(k-p)_\mu (k-p)_\nu}{M^2}$

$$\bar{u}(p')\left[(\not{k}-\not{p})(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)(\not{k}-\not{p})\right]u(p)$$
$$= \bar{u}(p')(\not{k}+\not{q}-m)(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)(\not{k}-m)u(p)$$
$$= \bar{u}(p')\left((k+q)^2-m^2\right)\gamma^\mu\left(k^2-m^2\right)u(p)$$
$$= \left((k+q)^2-m^2\right)(k^2-m^2)\,\bar{u}(p')\gamma^\mu u(p).$$

So this only contributes to $F_1(q^2)$.

Then,

$$\delta F_2 = \epsilon^2 \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2 z(1-z)}{m^2(1-z)^2 - xy q^2 + zM^2}$$

$$\delta a = \delta F_2(q^2=0) = \epsilon^2\frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\cdot\frac{2z(1-z)\cdot\frac{m^2}{M^2}}{z+(1-z)^2\cdot\frac{m^2}{M^2}}$$ where $m = m_e$ or m_μ

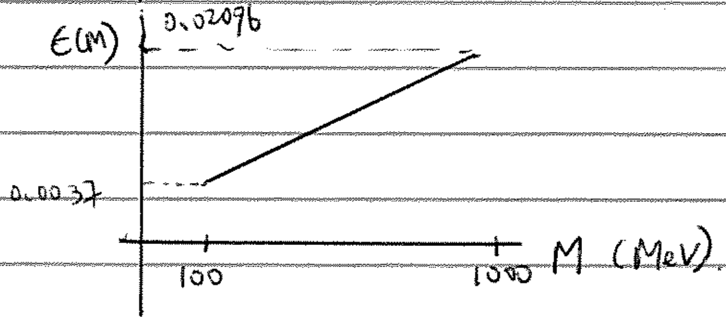
(b) For electron $m_e \ll M$

$$\delta a_e \cong \epsilon^2\frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\frac{2z(1-z)}{z}\frac{m_e^2}{M^2} = \frac{\epsilon^2\alpha}{3\pi}\frac{m_e^2}{M^2}$$

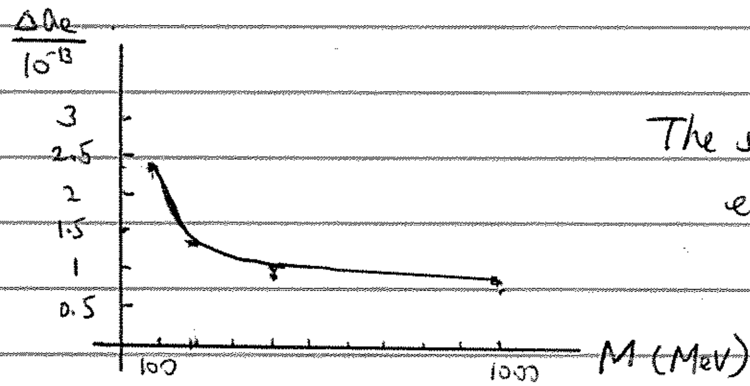
For muon, $M \sim m_\mu$

$$\delta a_\mu \cong \epsilon^2\frac{\alpha}{2\pi}\int_0^1 dz\,\frac{2z(1-z)^2}{z+(1-z)^2} = \frac{\epsilon^2\alpha}{\pi}\underbrace{\left(\frac{\pi}{3\sqrt{3}}-\frac{1}{2}\right)}_{\approx 0.1}$$

(c) Required ϵ for M s.t. $\Delta a_\mu = 3\times 10^{-9}$ $\quad \epsilon(M) = \sqrt{\dfrac{\Delta a_\mu}{\frac{\alpha}{2\pi} I(M)}}$, $I(M) = \int_0^1 \dfrac{2z(1-z)^2}{(1-z)^2+\left(\frac{M}{m_\mu}\right)^2 z}$



(d) We plug part (c) into $\Delta a_e \cong \dfrac{\epsilon^2\alpha}{3\pi}\dfrac{m_e^2}{M^2}$ where $\epsilon = \epsilon(M)$ in part (c).



The shift is smaller than current experimental error.