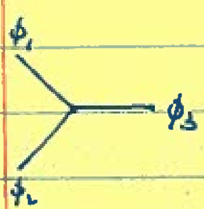
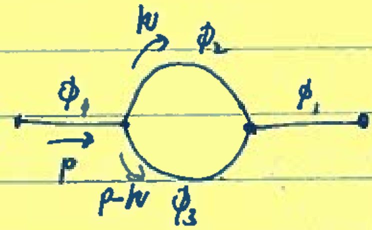
Problem Set 3 – Solutions

1. (a) 

$= -i\lambda$



$$-i\Pi(p^2) = (-i\lambda)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m_2^2+i\epsilon} \cdot \frac{i}{(p-k)^2-m_3^2+i\epsilon}$$

$$\Pi(p^2) = i\lambda^2 \int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{k^2-m_2^2+i\epsilon} \cdot \frac{1}{(p-k)^2-m_3^2+i\epsilon}$$

$$= i\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{\{x(k^2-m_2^2)+(1-x)[(p-k)^2-m_3^2]\}^2}$$

$$= i\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{\{k^2-2(1-x)(p\cdot k)-m_2^2x+(1-x)(p^2-m_3^2)\}^2}$$

$$= i\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{\{[k-(1-x)p]^2-(1-x)^2p^2-m_2^2x+(1-x)(p^2-m_3^2)\}^2}$$

Let $k' = k-(1-x)p$

$$= i\lambda^2 \int_0^1 dx \int \frac{d^dk'}{(2\pi)^d} \cdot \frac{1}{[k'^2-(m_2^2x+(1-x)m_3^2-x(1-x)p^2)]^2}$$

Use (A.44) in Peskin (P807)
$$\int \frac{d^d\ell}{(2\pi)^d} \cdot \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

and (A.52) w/ $2\epsilon = 4-d$
$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + O(\epsilon)\right)$$

$$\Pi(p^2) = i\lambda^2 \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \cdot \left(\frac{1}{m_2^2x+m_3^2(1-x)-p^2x(1-x)}\right)^{2-\frac{d}{2}}$$

$$= -\lambda^2 \int_0^1 dx \cdot \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon} - \log\left(m_2^2x+m_3^2(1-x)-p^2x(1-x)\right) - \gamma + \log(4\pi)\right]$$

$$= \frac{\lambda^2}{(4\pi)^2}\left[\int_0^1 dx \log\left(m_2^2x+m_3^2(1-x)-p^2x(1-x)\right)\right] - (m_1^2-m_{1R}^2).$$

When $p^2 < m_2^2 < m_3^2$

$$m_2^2 x + m_3^2(1-x) - p^2 x(1-x) > p^2 x + p^2(1-x) - p^2 x(1-x) = p^2\left(\left(x-\tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right) > 0$$

Thus $\Pi(p^2)$ is real.

(b) The physical mass M_1 is given by the pole of $\dfrac{1}{p^2 - m_1^2 - \Pi(p^2)}$

$$p^2 - m_1^2 - \Pi(p^2) = p^2 - m_{1R}^2 - \lambda^2 F(p^2)$$

where $F(p^2) = \frac{1}{(4\pi)^2}\int_0^1 dx \log\left(m_2^2 x + m_3^2(1-x) - p^2 x(1-x)\right)$

So $\quad M_1^2 - m_{1R}^2 - \lambda^2 F(M_1^2) = 0$

Compute shift $\quad s = M_1 - m_{1R} = O(\lambda^2)$

$$(s + m_{1R})^2 - m_{1R}^2 - \lambda^2 F\left((s+m_{1R})^2\right) = 0$$

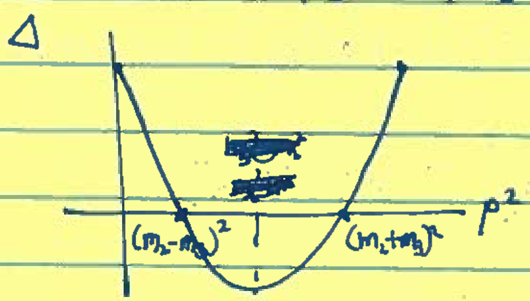
$$\underbrace{s^2}_{O(\lambda^4)} + 2m_{1R}s = \lambda^2 F(m_{1R}^2) + O(\lambda^4)$$

$$s = \frac{\lambda^2}{2m_{1R}} F(m_{1R}^2) + O(\lambda^4)$$

Evaluate $F(p^2)$: $\quad \int \log(Ax^2 + Bx + C) = \frac{1}{A}\sqrt{B^2-4AC}\,\tan^{-1}\left(\frac{2Ax+B}{\sqrt{B^2-4AC}}\right) - 2x + \frac{2Ax+B}{2A}\log(Ax^2+Bx+C)$

Let $\Delta = B^2 - 4AC$. $\quad F(p^2) = \frac{1}{(4\pi)^2}\int_0^1 dx \log\Big[\underbrace{p^2}_{A} x^2 + \underbrace{(m_2^2 - m_3^2 - p^2)}_{B} x + \underbrace{m_3^2}_{C}\Big] dx$

$$\Delta = (m_2^2 - m_3^2 - p^2)^2 - 4p^2 m_3^2 = \left[p^2 - (m_2+m_3)^2\right]\left[p^2 - (m_2-m_3)^2\right]$$



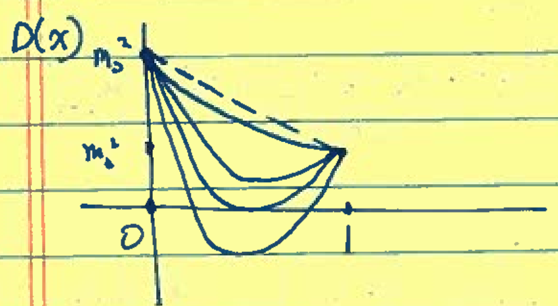
When $p^2 < m_2 < m_3$

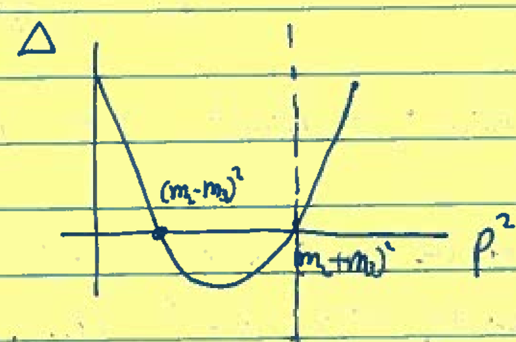
$$F(p^2) = \frac{1}{(4\pi)^2}\left[\frac{\sqrt{|\Delta|}}{p^2}\left(\tan^{-1}\left(\frac{m_2^2-m_3^2+p^2}{\sqrt{|\Delta|}}\right)+\tan^{-1}\left(\frac{m_3^2-m_2^2+p^2}{\sqrt{|\Delta|}}\right)\right)\right.$$
$$\left. + \frac{p^2+m_2^2-m_3^2}{2p^2}\log(m_2^2) + \frac{p^2+m_3^2-m_2^2}{2p^2}\log(m_3^2) - 2\right]$$
$$= \frac{1}{(4\pi)^2}\left[\frac{\sqrt{|\Delta|}}{p^2}\left(\tan^{-1}\left(\frac{m_2^2-m_3^2+p^2}{\sqrt{|\Delta|}}\right)+\tan^{-1}\left(\frac{m_3^2-m_2^2+p^2}{\sqrt{|\Delta|}}\right)\right) + \log(m_2 m_3) + \frac{1}{p^2}(m_2^2-m_3^2)\log\left(\frac{m_2}{m_3}\right) - 2\right]$$

(c) Let $D = Ax^2 + Bx + C$.

$D(x=0) = m_3^2$

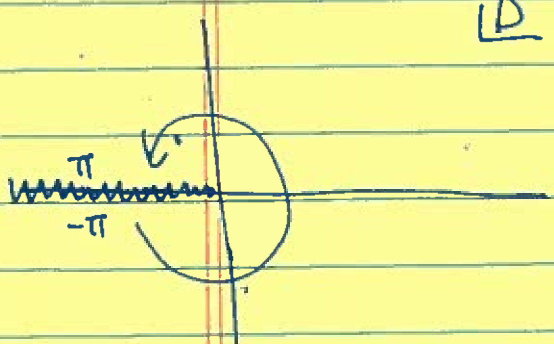
$D(x=1) = m_2^2$





Observe that $D(x)$ first develops zeros and negative values on the interval $x \in [0,1]$ for $p^2 > (m_2+m_3)^2$.

$F(p^2) = \frac{1}{(4\pi)^2}\int_0^1 dx \log(D(x))$ has branch cut for $p^2 > (m_2+m_3)^2$



Put branch cut of $\log D$ on $(-\infty, 0]$ on complex D-plane.

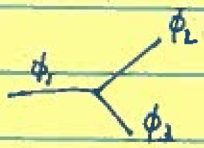
$\log D = \log|D| + i \arg D$.

$\arg D$ goes from $-\pi$ to π on chosen branch.

$$D(p^2+i\epsilon) = D(p^2) - i\epsilon\underbrace{x(1-x)}_{\text{positive on } (0,1)} = D(p^2) - i\epsilon'$$

Thus at $p^2+i\epsilon$ along the cut, $\log D = \log|D| - i\pi \Rightarrow \mathrm{Im}\,\Pi(p^2)$ negative

at $p^2-i\epsilon$ along the cut, $\log D = \log|D| + i\pi \Rightarrow \mathrm{Im}\,\Pi(p^2)$ positive

(d) 

$= -i\lambda$. $\quad |M|^2 = \lambda^2$

In m_1 COM frame, $p_1^\mu = (m_1, 0)$,

$p_1^\mu = p_2^\mu + p_3^\mu \qquad p_2^\mu = (\sqrt{m_2^2+\vec{p}^{\,2}}, \vec{p})$

$\qquad\qquad\qquad\qquad p_3^\mu = (\sqrt{m_3^2+\vec{p}^{\,2}}, -\vec{p})$

$$m_1 = \sqrt{m_3^2+\vec{p}^{\,2}} + \sqrt{m_2^2+\vec{p}^{\,2}} \Rightarrow \vec{p}^{\,2} = \frac{[m_1^2-(m_2-m_3)^2][m_1^2-(m_2+m_3)^2]}{4m_1^2}$$

For one particle decaying into 2 particles.

decay ~~width~~ $\Gamma = \frac{1}{2m_1}\int_{\text{all final states}} |M_{fi}|^2 \cdot \sigma$, with $\sigma = \frac{1}{16\pi^2}\frac{|\vec{p}|\,d\Omega}{E_{tot}}$, $\quad E_{tot} = m_1$, $\int d\Omega = 4\pi$

$$\therefore \Gamma = \frac{1}{2m_1}\cdot\frac{1}{16\pi^2}\frac{|\vec{p}|(4\pi)\lambda^2}{m_1} \cancel{= \frac{\lambda^2}{16m_1^3\pi}\sqrt{[m_1^2-(m_2-m_3)^2][m_1^2-(m_2+m_3)^2]}}$$

$$= \frac{\lambda^2}{16\pi m_1^3}\sqrt{[m_1^2-(m_2-m_3)^2][m_1^2-(m_2+m_3)^2]}$$

(e) $$\frac{i}{p^2 - m_1^2 - \Pi(p^2)} = \frac{i}{p^2 - M_1^2 - i\,\mathrm{Im}\,\Pi(p^2)} = \frac{i}{p^2 - M_1^2 - i\lambda^2\,\mathrm{Im}\,F(p^2)}$$

$\mathrm{Im}\,F(p^2) = \frac{1}{(4\pi)^2}(-i\pi)(x_b - x_a)$ where $x_a < x_b$ are the zeros of $D(x)$ inside interval $[0,1]$:

$$x_{a,b} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

so $$x_b - x_a = \frac{\sqrt{B^2-4AC}}{A} = \frac{\sqrt{[p^2-(m_2+m_3)^2][p^2-(m_2-m_3)^2]}}{p^2}$$

So the pole is displaced from $p^2 = M_1^2$ by an ~~imp~~ imaginary amount of $O(\lambda^2)$

$$\cancel{F(p^2)} \; -i\lambda^2 F(M_1^2 + iO(\lambda^2)) = -i\lambda^2 F(M_1^2) + O(\lambda^4).$$

so we can replace $F(p^2)$ by $F(M_1^2)$ here. In fact, can replace by $F(m_1^2)$ by same logic.

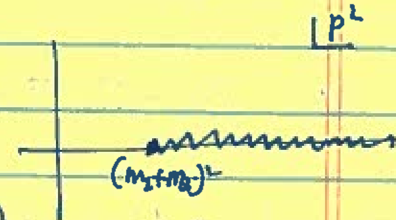
From Peskin (4.64). near the pole

$$\text{propagator} = \frac{1}{p^2 - M_1^2 + \frac{i\lambda^2}{16\pi}\cdot\frac{\boxed{\sqrt{[m_1^2-(m_2+m_3)^2][m_1^2-(m_2-m_3)^2]}}}{m_1^2}} = \frac{1}{2E_p\left(p^0 - E_p + i\frac{M_1}{E_p}\cdot\frac{1}{2M_1}\cdot\frac{\lambda^2}{16\pi}\cdot\frac{\sqrt{\cdots}}{m_1^2}\right)}$$

$$= \frac{1}{2E_p\left(p^0 - E_p + i\frac{M_1}{E_p}\cdot\frac{\Gamma}{2}\right)}$$ where Γ is the decay width from part (d)

(f) This follows from our discussion in part (c). The latter pole corresponds to incorrect time-ordering and thus is irrelevant.

As we saw in (c), at $p^2+i\epsilon$ above the cut, $\text{Im}\,\Pi(p^2)<0$; and at $p^2-i\epsilon$ below the cut, $\text{Im}\,\Pi(p^2)>0$. This shows that the propagator has an extensive first Riemann sheet w/ branch cut along $p^2>(m_2+m_3)^2$ real axis and no other singularities on this sheet.



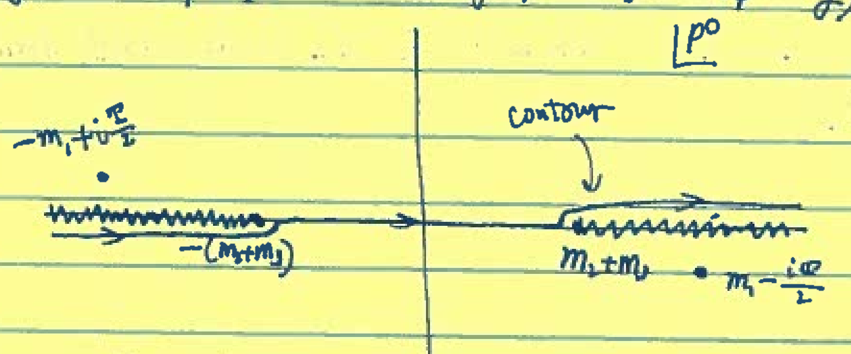
If we start from above the cut and analytically continue across the cut to below the cut, we are on the 2nd Riemann sheet and this is the pole we find in part (e).

Since $\Pi(p^2) = [\Pi(p^{2*})]^*$, the situation is symmetric; if we start from below the cut, we can analytically contine ~~and~~ across and above the cut and get a different pole on a different 2nd Riemann sheet.

The 1st pole is at $p^2=(M_1-i\frac{\Gamma}{2})^2$, 2nd pole at $p^2=(M_1+i\frac{\Gamma}{2})^2$.

Consider $\langle T\phi_1(x)\phi_1(y)\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-\cancel{m^2}-\Pi(p^2)}$

$t=x^0-y^0$. In p^0 plane (setting $\vec{p}=0$ for simplicity), we need to evaluate



$$\int \frac{dp^0}{2\pi} e^{-ip^0 t} \frac{i}{p^{0\,2}-(m_1-\frac{i\Gamma}{2})^2}$$

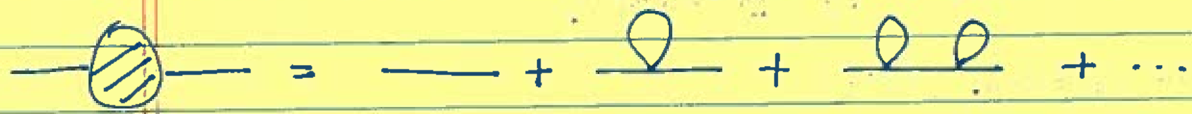
$$=\int \frac{dp^0}{2\pi} e^{-ip^0 t} \frac{i}{\left[p^0-(m_1-\frac{i\Gamma}{2})\right]\left[p^0-(-m_1+i\frac{\Gamma}{2})\right]}$$

For $t>0$, we close the contour in the lower-half plane and pick up the residue at the pole $m_1-\frac{i\Gamma}{2}$; for $t<0$, we close the contour in upper-half plane and pick up residue at $-m_1+i\frac{\Gamma}{2}$. In both cases we have pushed the contour onto 2nd Rieman sheet containing the pole at $p^2=(M_1-i\frac{\Gamma}{2})^2$. The pole at $p^2=(M_1+i\frac{\Gamma}{2})^2$ is never picked up, and so is irrelevant by causality.

2. (a) $iM =$  $= (4!)\left(\frac{-i\lambda_0}{4!}\right) = -i\lambda_0$

(b) $-iM^2(p^2) =$ 

Then if we compute the geometric sum of this diagram, i.e,

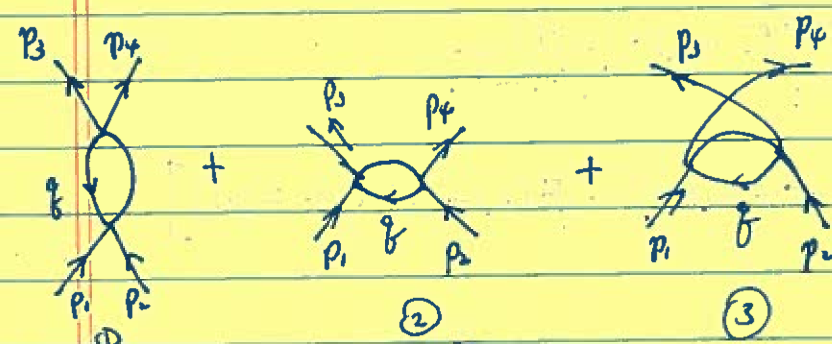


$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-iM^2)\cdot\frac{i}{p^2-m_0^2} + \cdots$$

$$= \frac{i}{p^2-m_0^2}\cdot\frac{1}{1-\frac{M^2}{p^2-m_0^2}} = \frac{i}{p^2-m_0^2-M^2(p^2)}.$$

which means the effect of this set of diagrams is to shift to physical mass $m^2 = m_0^2 + M^2$

(c)



iM in order $O(\lambda_0^2) = \frac{(-i\lambda_0)^2}{2}$.

Define

① $p = p_1 + p_2$

② $p = p_1 - p_3$

③ $p = p_1 - p_4$

Then all three diagrams are of the form

$$(-i\lambda_0)^2\, iV(p^2) = \frac{1}{2}\cancel{(-i\lambda_0)^2}\int\frac{d^4q}{(2\pi)^4}\cdot\frac{i}{q^2-m^2}\cdot\frac{i}{(p+q)^2-m^2}$$

Then overall in order $O(\lambda_0^2)$

$$iM = (-i\lambda_0)^2\cdot\left[iV(s) + iV(t) + iV(u)\right]$$

(d) $$iV(p^2) = -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\cdot\frac{1}{q^2-m^2}\cdot\frac{1}{(p+q)^2-m^2}$$

$$= -\frac{1}{2}\int_0^1 dx \int\frac{d^dq}{(2\pi)^d}\cdot\frac{1}{\left(x(q^2-m^2)+(1-x)[(p+q)^2-m^2]\right)^2}$$

$q' = q+(1-x)p$

$$= -\frac{1}{2}\int_0^1 dx\int\frac{d^dq'}{(2\pi)^d}\cdot\frac{1}{\left(q'^2-[m^2-x(1-x)p^2]\right)^2}$$

Use (A.44), (A.52) in Peskin (p807).

$$iV(p^2) = -\frac{1}{2}\int_0^1 dx\,\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(\epsilon)}{\Gamma(2)}\left(\frac{1}{m^2-x(1-x)p^2}\right)^{2-\frac{d}{2}}$$

$$= -\frac{i}{32\pi^2}\int_0^1 dx\left[\frac{1}{\epsilon}-\gamma+\log(4\pi)-\log(m^2-x(1-x)p^2)\right]$$

$$= -\frac{i}{32\pi^2}\left[\frac{1}{\epsilon}-\gamma+\log(4\pi)\right]+\frac{i}{32\pi^2}\underbrace{\int_0^1 dx\,\log(m^2-x(1-x)p^2)}_{F(p^2)}$$

$$F(p^2) = -2+2\log(m)+\frac{\sqrt{4m^2-p^2}}{|p|}\tanh^{-1}\left(\frac{|p|}{\sqrt{4m^2-p^2}}\right)$$

(e) $$iM = -i\lambda_0+\frac{i\lambda_0^2}{32\pi^2}\left[3\left(\frac{1}{\epsilon}-\gamma+\log(4\pi)\right)-\left(F(s)+F(t)+F(u)\right)\right]$$

(f) At $s=4m^2$ and $t=u=0$,

$$iM = -i\lambda_0+\frac{i\lambda_0^2}{32\pi^2}\left[3\left(\frac{1}{\epsilon}-\gamma+\log 4\pi-\log m^2\right)-\int_0^1 dx\,\log(1-4x(1-x))\right]$$

$= -i\lambda$ by definition.

Then,

$$\lambda_0 = \lambda+\frac{\lambda^2}{32\pi^2}\left[3\left(\frac{1}{\epsilon}-\gamma+\log 4\pi-\log m^2\right)-\int_0^1 dx\,\log(1-4x(1-x))\right]+O(\lambda^3)$$

(g). $iM = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\frac{m^2-x(1-x)s}{m^2-4x(1-x)m^2} + \log\frac{m^2-x(1-x)t}{m^2} + \log\frac{m^2-x(1-x)u}{m^2}\right] + O(\lambda^3)$

Note all $\frac{1}{\epsilon}$'s cancel out.

(h) $\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|M|^2}{64\pi^2 E_{cm}^2}$ (4.85) on p107 Peskin.

$$= \frac{1}{64\pi^2 s}\lambda^2$$

$$\sigma(s) = \underbrace{\frac{1}{2}}_{\text{identical final particles}} \underbrace{\int d\Omega}_{4\pi} \left(\frac{d\sigma}{d\Omega}\right) = \frac{\lambda^2}{32\pi s}$$

When $m^2 - x(1-x)s < 0$ $\quad \frac{1-\sqrt{1-\frac{4m^2}{s}}}{2} < x < \frac{1+\sqrt{1-\frac{4m^2}{s}}}{2}$

$$\log\left[m^2 - x(1-x)(s+i\epsilon)\right] = \log\left[x(1-x)s - m^2\right] - i\pi$$

$$\log\left[m^2 - x(1-x)(s-i\epsilon)\right] = \log\left[x(1-x)s - m^2\right] + i\pi$$

$$M(s+i\epsilon) - M(s-i\epsilon) = -\frac{\lambda^2}{32\pi^2}(-2\pi i)\sqrt{1-\frac{4m^2}{s^2}}$$

$$\text{Im } M = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}$$

We indeed have $\text{Im } M = s\sqrt{1-\frac{4m^2}{s}}\,\sigma(s)$