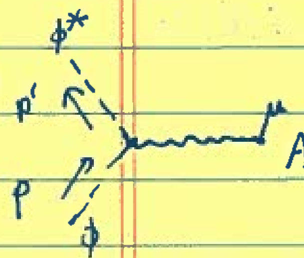
Problem Set 2 - Solutions

$1.^{(a)}\ (D_\mu\phi)^*(D^\mu\phi) = (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial^\mu\phi + ieA^\mu\phi)$

$$= \partial_\mu\phi^*\partial^\mu\phi + ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2 A_\mu A^\mu\phi^*\phi$$

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, $\quad \mathcal{L}_0 = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$

$$e^{i\int\mathcal{L}} = e^{i\int\mathcal{L}_0+\mathcal{L}_{int}} \cong e^{i\int\mathcal{L}_0}\Big[1 \underbrace{-eA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)}_{①} + \underbrace{ie^2A_\mu A_\nu g^{\mu\nu}\phi^*\phi}_{②}\Big]$$

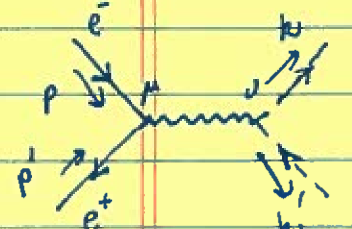
① 

$\partial_\mu\phi \sim -ip_\mu\phi$, $\partial_\mu\phi^* \sim ip_\mu\phi^*$

A^μ ① vertex $= -e(ip'^\mu + ip^\mu) = -ie(p+p')^\mu$

② 

$= 2ie^2 g^{\mu\nu}$ The factor of 2 comes from the interchange of the two photon lines.

(b) 

$q = p + p' = k + k'$

set $m_e = 0$

$$i\mathcal{M} = \bar v^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)(k-k')^\nu$$

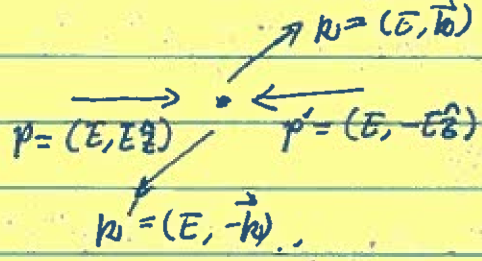
$$\frac14\sum_{s,s'}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\left(\bar u^s(p)\gamma^\mu u^{s'}(p')\bar v^{s'}(p')\gamma^\nu u^s(p)\right)(k'-k)_\mu(k'-k)_\nu$$

$$= \frac{e^4}{4q^4}\mathrm{Tr}(\not p\gamma^\mu\not p'\gamma^\nu)(k'-k)_\mu(k'-k)_\nu$$

$$= \frac{e^4}{\cancel{4}q^4}\times\cancel{4}\left[p^\mu p'^\nu + p^\nu p'^\mu - (p\cdot p')g^{\mu\nu}\right](k'-k)_\mu(k'-k)_\nu$$

$$= \frac{e^4}{q^4}\left[2\big(p\cdot(k'-k)\big)\big(p'\cdot(k'-k)\big) - (p\cdot p')(k'-k)^2\right]$$

Go to center of mass frame



$|\vec{k}| = \sqrt{E^2 - m^2}$, $\vec{k}\cdot\hat{z} = |\vec{k}|\cos\theta$

total energy $E_{COM} = 2E$

$q^2 = (p+p')^2 = 4E^2$. $\qquad p\cdot(k'-k) = 2(\vec{k}\cdot\hat{z})E = 2E|\vec{k}|\cos\theta$

$p\cdot p' = 2E^2 \qquad p'\cdot(k'-k) = -2E|\vec{k}|\cos\theta$.

$(k'-k)^2 = -4|\vec{k}|^2$.

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{16E^4}\left[2(-2E|\vec{k}|\cos\theta)(2E|\vec{k}|\cos\theta) - 2E^2(-4|\vec{k}|^2)\right]$$

$$= \frac{e^4}{2}\left(1-\frac{m^2}{E^2}\right)\sin^2\theta.$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{COM} = \frac{1}{2E_{COM}^2}\cdot\frac{|\vec{k}|}{16\pi^2 E_{COM}}\cdot\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

$$= \frac{\alpha^2}{8E_{cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}\sin^2\theta$$

$$\sigma_{total} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \sin\theta \frac{d\sigma}{d\Omega}$$

$$= \frac{\pi\alpha^2}{3E_{cm}^2}\left(1-\frac{m^2}{E^2}\right)^{\frac{3}{2}}$$

(c) The generating functional is

$$Z[J, J^*] = \int D\phi\, D\phi^* \exp\left[i\int d^4x \left((\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial^\mu \phi + ieA^\mu \phi) - m^2\phi^*\phi + J^*\phi + J\phi^*\right)\right]$$

Let $\phi^* \to \phi^* + \delta\phi^*$ which leaves $D\phi$, $D\phi^*$ invariant

$$0 = \delta Z = \int D\phi\, D\phi^* \exp\left[i\int d^4y \left((\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial^\mu \phi + ieA^\mu \phi) - m^2\phi^*\phi + J^*\phi + J\phi^*\right)\right]$$
$$\times\, i\int d^4x\, \delta\phi^*(x)\left[\left(\underset{\uparrow}{(-\partial_\mu} - ieA_\mu)(\partial^\mu + ieA^\mu)\phi(x) - m^2\phi(x) + J(x)\right]$$

Integration by parts

Take functional derivative $\frac{\delta}{\delta J(y)}$ at $J = J^* = 0$

recall
$$\frac{\delta}{\delta J(y)}\int d^4x\, J(x)\phi(x) = \phi(y) = \int d^4x\, \phi(x)\,\delta(x-y)$$

$$0 = \int D\phi\, D\phi^*\, e^{iS}\, i\int d^4x\, \delta\phi^*(x)\left[(-D_\mu D^\mu - m^2)\phi(x)\,\phi^*(y) + \delta(x-y)\right]$$

$$\Rightarrow \underbrace{(D_\mu D^\mu + m^2)}_{\text{with respect to } x} \langle\Omega|\, T\phi(x)\,\phi^*(y)\,|\Omega\rangle = -i\,\delta^{(4)}(x-y)$$

(d)

$$ej^\mu = \frac{\delta \mathcal{L}}{\delta A_\mu} = \partial_\nu F^{\nu\mu} - ie\phi^* D^\mu\phi + ie\phi(D^\mu\phi)^*$$

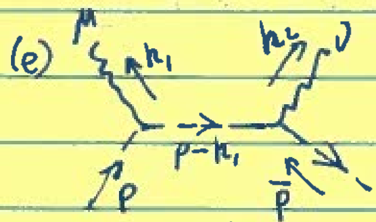
$$\Rightarrow j^\mu = \frac{1}{e}\partial_\nu F^{\nu\mu} - i\phi^*\partial^\mu\phi + i\phi\,\partial^\mu\phi^* + 2eA^\mu\phi^*\phi$$

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \quad \text{since} \quad F^{\mu\nu} = -F^{\nu\mu}$$

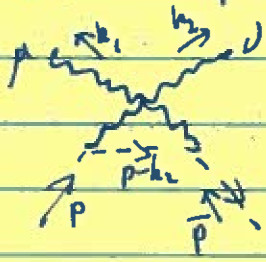
$$\partial_\mu j^\mu = -i\phi^*\partial^2\phi + i\phi\,\partial^2\phi^* + 2e\,\partial_\mu A^\mu\phi^*\phi + 2e\phi A^\mu\partial_\mu\phi^* + 2e\phi^* A^\mu\partial_\mu\phi$$

Note that
$$D_\mu D^\mu\phi = (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi$$
$$= \partial^2 + 2ie A\cdot\partial\phi + ie\phi(\partial\cdot A) - e^2A^2\phi$$

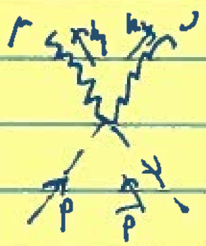
$$\Rightarrow \partial_\mu j^\mu = -i\phi^*(D^2+m^2)\phi + ie\,\phi\,(D^2+m^2)\phi^* = 0$$

(e) 

$$= -ie(p+\bar{p}-k_1)^\mu \epsilon_\mu^*(k_1)\,\frac{i}{(p-k_1)^2-m^2+i\epsilon}\,(-ie)(p-k_1-\bar{p})^\nu \epsilon_\nu^*(k_2)$$



$$= -ie(p+\bar{p}-k_2)^\nu \epsilon_\nu^*(k_2)\,\frac{i}{(p-k_2)^2-m^2+i\epsilon}\,(-ie)(p-k_2-\bar{p})^\mu \epsilon_\mu^*(k_1)$$



$$= 2ie^2 g^{\mu\nu}\epsilon_\mu^*(k_1)\,\epsilon_\nu^*(k_2)$$

$$iM = -ie^2\epsilon_\mu^*(k_1)\,\epsilon_\nu^*(k_2)\left[\frac{(2p-k_1)^\mu(p-k_1-\bar{p})^\nu}{(p-k_1)^2-m^2} + \frac{(p-k_2-\bar{p})^\mu(2p-k_2)^\nu}{(p-k_2)^2-m^2} - 2g^{\mu\nu}\right]$$

So $$k_{1\mu}M^{\mu\nu} = -e^2\left[\frac{2p\cdot k_1 - k_1^2}{(p-k_1)^2-m^2}(p-k_1-\bar{p})^\nu + \frac{k_{1\mu}(p-k_2-\bar{p})^\mu}{(p-k_2)^2-m^2}(2p-k_2)^\nu - 2k_1^\nu\right]$$

Note $$(p-k_1)^2-m^2 = k_1^2-2p\cdot k_1$$

$$(p-k_2)^2-m^2 = (\bar{p}-k_1)^2-m^2 = k_1^2-2\bar{p}\cdot k_1$$

$$k_{1\mu}(p-k_2-\bar{p})^\mu = k_{1\mu}(k_1-2\bar{p})^\mu = k_1^2-2\bar{p}\cdot k_1$$

Thus $$k_{1\mu}M^{\mu\nu} = -e^2\left(-p+k_1+\bar{p}+2p-k_2-2k_1\right)^\nu = 0.$$

Note we didn't use $k_1^2=0$ or $k_2^2=0$, so A_μ can be off-shell.

2. (a) First perform integral over B field

$$\int DB \, \exp\left[i\int d^4x \left(\frac{\xi}{2}B^2 + B\partial_\mu A^\mu\right)\right] = \int DB \, \exp\left[i\int d^4x \left[\frac{\xi}{2}\left(B+\frac{1}{\xi}\partial_\mu A^\mu\right)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]\right]$$

$$= f(\xi)\,\exp\left[i\int d^4x \left(-\frac{1}{2\xi}\right)(\partial_\mu A^\mu)^2\right]$$

Now $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}\partial_\mu A^\mu \partial_\nu A^\nu$$

Integrate by parts, this is $= -\frac{1}{4}\left(-A^\nu\partial^2 A_\nu + A_\nu\partial^\mu\partial^\nu A_\mu + A^\mu\partial_\nu\partial_\mu A^\nu - A_\mu\partial^2 A_\mu\right)$

$$+\frac{1}{2\xi}A^\mu\partial_\mu\partial_\nu A^\nu$$

$$= \frac{1}{2}A_\mu\left[\partial^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu$$

(b) Note $\delta\psi = -ie\epsilon c\psi$, $\delta\bar\psi = ie\bar\psi\epsilon c$

$$\delta F_{\mu\nu} = \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu) = \epsilon(\partial_\mu\partial_\nu c - \partial_\nu\partial_\mu c) = 0.$$

$\delta\left(\frac{B^2}{2}\right) = 0$ because $\delta B = 0$.

$$\delta\left(B\partial_\mu A^\mu + \bar c(-\partial^2)c\right) = B\partial_\mu(\epsilon\partial^\mu c) - \epsilon B\partial^2 c = 0.$$

$$\delta\left(\bar\psi(i\gamma^\mu D_\mu - m)\psi\right) = \bar\psi\left(i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right)(-ie\epsilon c\psi)$$

$$+\bar\psi\left(i\gamma^\mu(ie)\epsilon\partial_\mu c\right)\psi$$

$$+ ie\bar\psi\epsilon c\left(i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right)\psi$$

$$= -ie\bar\psi\epsilon c(i\not{D} - m)\psi + ie\bar\psi\epsilon c(i\not{D} - m)\psi = 0$$

(c) $\delta^2 A_\mu = \epsilon \partial_\mu (\delta c) = 0$

$\delta^2 \psi = -ie\epsilon \big((\delta c)\psi + c(\delta\psi) \big) = -ie\epsilon c \, (-ie\epsilon c)\psi = 0$ since $c^2 = 0$

$\delta^2 c = 0$, $\delta^2 \bar{c} = \epsilon \, \delta B = 0$, $\delta^2 B = \delta(\delta B) = 0$