Problem Set #1 - Solutions

1. (a) $Z = \int_{-\infty}^{\infty} d\phi\, e^{-\phi^2/2A} = \sqrt{2\pi A}$

$$\langle\phi^n\rangle = \frac{1}{Z}\int_{-\infty}^{\infty} d\phi\, \phi^n\, e^{-\phi^2/2A} = \frac{1}{Z}\int_{-\infty}^{\infty} d\phi\, e^{-\alpha\phi^2}\phi^n, \quad \alpha = \frac{1}{2A}.$$

For n odd, by $\phi \to -\phi$ symmetry $\langle\phi^n\rangle = 0$

For n even,

consider $\frac{\partial}{\partial\alpha}\left(\int_{-\infty}^{\infty} e^{-\alpha\phi^2}\phi^n d\phi\right) = -\int_{-\infty}^{\infty}\phi^{n+2}e^{-\alpha\phi^2}d\phi$

so
$$\int_{-\infty}^{\infty}\phi^n e^{-\alpha\phi^2}d\phi = (-1)^{\frac{n}{2}}\left(\frac{\partial}{\partial\alpha}\right)^{\frac{n}{2}}\left(\int_{-\infty}^{\infty}e^{-\alpha\phi^2}d\phi\right)$$
$$= (-1)^{\frac{n}{2}}\left(\frac{\partial}{\partial\alpha}\right)^{\frac{n}{2}}\left(\frac{\sqrt{\pi}}{\sqrt{\alpha}}\right)$$
$$= (-1)^{\frac{n}{2}}\sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-\left(\frac{n}{2}-1\right)\right)\alpha^{-\frac{1+n}{2}}$$
$$= (n-1)!!\, A^{\frac{n}{2}}$$

So
$$\mathcal{N}(n) = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even}\end{cases}$$

(b) $[\phi\,\phi] = \langle\phi^2\rangle = A$

For $\langle\phi^n\rangle$, to see $\mathcal{N}(n)$ is the number of contractions, we put all n ϕ's in a row. First step we choose the ϕ that contracts with the first one in the row, and there are $(n-1)$ choices. Delete these two and continue this procedure, at the end the number of contractions is $(n-1)!! = \mathcal{N}(n)$.

(c) A is real-symmetric $\Rightarrow$ $A = ODO^T$, where D is diagonal and real, and O is orthogonal, $O^T = O^{-1}$. Let $D = \mathrm{diag}(d_1, \cdots, d_n)$.

Let $\varphi = O\phi$, then
$$Z = \int_{-\infty}^{\infty} d^n(O\phi)\, e^{-\frac{1}{2}\sum_{i=1}^{n}\varphi_i d_i^{-1}\varphi_i} = \prod_{i=1}^{n}\int_{-\infty}^{\infty}d\varphi_i\, e^{-\frac{\varphi_i^2}{2d_i}}$$
$$= (2\pi)^{\frac{n}{2}}(\det A)^{\frac{1}{2}}.$$

$\langle \phi_i \phi_j \rangle = O_{ki} \langle \varphi_k \varphi_l \rangle O_{lj}$

$\langle \varphi_k \varphi_l \rangle = \delta_{kl} d_k$, so $\langle \phi_i \phi_j \rangle = (O^T D O)_{ij} = A_{ij}$

Lastly, $\langle \phi_i \phi_j \cdots \phi_k \rangle = O_{mi} O_{nj} \cdots O_{lk} \langle \varphi_m \varphi_n \cdots \varphi_l \rangle$

From previous parts, $\langle \varphi_m \varphi_n \cdots \varphi_l \rangle$ equals the sum of all contractions with each contracted pair giving rise to one eigenvalue d_i. Together with factors of O, we see that $\langle \phi_i \phi_j \cdots \phi_k \rangle$ is equal to the sum of all possible contractions with contraction of $\phi_i \phi_j$ giving rise to A_{ij}.

2. (a) $x(\tau) = \sum_n x_n \cdot \frac{1}{\sqrt{\beta}} e^{2\pi i n \tau/\beta}$

Since $x(\tau)$ is real, we have $x_n = x_{-n}^*$.

$$S_E = \frac{1}{2}\sum_{n=-\infty}^{\infty} x_n x_{-n}\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2}\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]|x_n|^2 + \frac{1}{2}\omega^2 x_0^2$$

So $Z = Tr\left(e^{-\beta H}\right) = \int_{x(0)=x(\beta)} Dx(\tau)\, e^{-S_E}$

$$= \prod_{n>0} \int dx_n\, e^{-\frac{1}{2}\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right] x_n x_n^*} \int dx^0 e^{-\frac{1}{2}\omega^2 x_0^2}$$

$$= \prod_{n>0}\left(\frac{2\pi}{\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2}\right)\left(\sqrt{\frac{2\pi}{\omega^2}}\right).$$

(b). Standard way for harmonic oscillator.

The spectrum is $E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$

So $Z = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega} = \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta\omega}} = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}$

On the other hand, from part (a).

using $\frac{1}{\sinh z} = \frac{1}{z}\prod_{n>0}\left(1+\frac{z^2}{(n\pi)^2}\right)^{-1}$

$$Z = \frac{\sqrt{2\pi}}{\omega}\prod_{n>0}(2\pi)\cdot \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2}\cdot \frac{1}{1+\left(\frac{\beta\omega}{2\pi n}\right)^2}$$

$$= \frac{\sqrt{2\pi}}{\omega}\prod_{n>0}\left[2\pi\left(\frac{\beta}{2\pi n}\right)^2\right]\cdot \frac{\beta\omega}{2\sinh\frac{\beta\omega}{2}}$$

$$= f(\beta)\frac{1}{\sinh\frac{\beta\omega}{2}}.$$

The ω- independent prefactor $f(\beta)$ can be absorbed into the definition of the path integral measure. Thus this agrees w/ result from standard way.

(c) $\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t)\, e^{i\vec{k}\cdot\vec{x}}$

Since $\phi(x,t)$ is real, we have $\phi_{-k} = \phi_k^*$

$$S_E = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\mu \phi)^2 + m^2\phi^2\right]$$

$$= \frac{1}{2}\int d\tau \frac{d^3k}{(2\pi)^3}\left[\dot{\phi}_{\vec{k}}\dot{\phi}_{-\vec{k}} + (\vec{k}^2+m^2)\,\phi_{\vec{k}}\phi_{-\vec{k}}\right]$$

Each $\vec{k}$ is a harmonic oscillator with frequency $\omega_k = \sqrt{k^2+m^2}$

So standard way $\Rightarrow$ $Z = \prod_k \frac{1}{2\sinh\frac{\beta\omega_k}{2}}$

Path integral :

$$Z = \int D\phi\, e^{-S} = \int D\phi_{\vec{k},n} \exp\left[-\frac{1}{2}\sum_n \int \frac{d^3k}{(2\pi)^3}\phi_{-\vec{k},-n}\left(\left(\frac{2\pi n}{\beta}\right)^2 + k^2 + m^2\right)\phi_{\vec{k},n}\right]$$

$= \prod_k Z_k$ where Z_k is the partition function from part (b) w/ $\omega_k = \sqrt{k^2+m^2}$

use (b)
$$= f(\beta)\prod_k \frac{1}{\sinh\left(\frac{\beta\omega_k}{2}\right)}$$

Again two methods agree up to β- dependent factor

(d) Assume $\tau_2 = 0$, $\beta > \tau_1 > 0$,

$$\begin{aligned} \langle \psi(\tau)\bar{\psi}(0)\rangle &= \mathrm{Tr}\left(e^{-\beta H}\psi(\tau)\psi^\dagger(0)\right) \\ &= \mathrm{Tr}\left(e^{-(\beta-\tau)H}\psi\, e^{-H\tau}\psi^\dagger\right) \\ &= \mathrm{Tr}\left(\psi^\dagger e^{-(\beta-\tau)H}\psi\, e^{-H\tau}\right) \\ &= \langle \bar{\psi}(\beta)\psi(\tau)\rangle \end{aligned}$$

But also $\langle \psi(\tau)\bar{\psi}(0)\rangle = -\langle\bar{\psi}(0)\psi(\tau)\rangle$

So we have $\bar{\psi}(\beta) = -\bar{\psi}(0)$.

(e) $H = \omega\bar{\psi}\psi$

Standard way: This is a two level system $E = 0, \omega$

$$Z = \sum_{n=0,1} e^{-\beta\omega n} = 1 + e^{-\beta\omega}$$

Functional integral: anti-periodic boundary condition in part (d) implies

$$\psi(\tau) = \sum_n \frac{1}{\sqrt{\beta}}\psi_n e^{\frac{(2n+1)\pi i\tau}{\beta}}$$

$$\bar{\psi}(\tau) = \sum_n \frac{1}{\sqrt{\beta}}\bar{\psi}_n e^{-\frac{(2n+1)\pi i\tau}{\beta}}$$

$$\begin{aligned} Z &= \int D\psi D\bar{\psi}\, e^{-\int_0^\beta d\tau(\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi)} \\ &= \prod_n \int d\psi_n d\bar{\psi}_n \exp\left[-\bar{\psi}_{-n}\left(\frac{(2n+1)\pi i}{\beta} + \omega\right)\psi_n\right] \\ &= \prod_n \left(\frac{(2n+1)\pi i}{\beta} + \omega\right) \\ &= \prod_{n\geq 0}\left(\frac{(2n+1)^2\pi^2}{\beta^2} + \omega^2\right) = \prod_{n\geq 0}\left(\frac{(2n+1)^2\pi^2}{\beta^2}\right)\prod_{n\geq 0}\left(1 + \frac{\beta^2\omega^2}{4\pi^2(n+\frac{1}{2})^2}\right) \end{aligned}$$

Using $\prod_{n\geq 0}\left(1 + \frac{z^2}{(n+\frac{1}{2})^2}\right) = \cosh\pi z$, we have $Z = f(\beta)\cosh\frac{\beta\omega}{2}$

which agrees w/ result from standard method, except note that functional integral method gives a ground state energy $E = -\frac{\omega}{2}$.