Physics 331 - Problem Set #6

Solutions

1.) For a scalar field in the adjoint representation of SU(N)

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g f^{abc} A_\mu^b \Phi^c$$

if we write

$$\underset{\sim}{\Phi} = \Phi^a t^a \qquad \text{(an } N\times N \text{ matrix)}$$

we can multiply the above by t^a and write this as:

$$D_\mu \underset{\sim}{\Phi} = \partial_\mu \underset{\sim}{\Phi} - ig\underbrace{[A_\mu^b t^b, \underset{\sim}{\Phi}]}_{= i f^{abc} t^a A^b \Phi^c}$$

The kinetic term for Φ is then:

$$\mathcal{L} = \frac{1}{2}(D_\mu \Phi^a)^2 = \text{tr}\,(D_\mu \underset{\sim}{\Phi})^2$$

the gauge boson masses come from the piece of this $\mathcal{L}$ that is quadratic in A_μ:

$$\mathcal{L} = \mathrm{tr}\left(-ig\,[A^b_\mu t^b, \langle \Phi \rangle]\right)^2 = -g^2 A^a_\mu A^b\, \mathrm{tr}\,[t^a,\langle\Phi\rangle][t^b,\Phi]$$

If $[t^a, \langle\Phi\rangle] = 0$, the corresponding A^a_μ is massless. These t^a generate the unbroken symmetries.

Now consider the two cases in the problem set:

<u>SU(5) with</u> $\langle \Phi \rangle = A \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix}$

the SU(5) generators can be written with the basis

$$\begin{pmatrix} t^a & \\ & \end{pmatrix} \qquad \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix}$$

$\uparrow$ t^a = 4×4 Hermitian traceless

$$\frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$$

The first two structures commute with $\langle\Phi\rangle$. These matrices generate $SU(4) \times U(1)$

The other two structures have nonzero commutators with $\langle \underset{\sim}{\Phi} \rangle$. For example:

$$\left[\frac{1}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \langle \underset{\sim}{\Phi} \rangle \right] = \frac{A}{2} \begin{pmatrix} & -4-1 \\ +4+1 & \end{pmatrix}$$

$$= (-5A) \cdot \frac{1}{2} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

then $$- \mathrm{tr} \left([t^a, \langle \underset{\sim}{\Phi} \rangle] [t^a, \underset{\sim}{\Phi}] \right) = \frac{25 A^2}{2}$$

$$\frac{1}{2} m_a^2 = - g^2 \, \mathrm{tr} \left([t^a, \langle \underset{\sim}{\Phi} \rangle] [t^a, \langle \underset{\sim}{\Phi} \rangle] \right)$$

$$= \frac{25 A^2}{2} \cdot g^2 \quad \Rightarrow \quad \underline{m_a = 5 g A}$$

$$\left[\frac{1}{2} \begin{pmatrix} & -i \\ i & \end{pmatrix}, \langle \underset{\sim}{\Phi} \rangle \right] = \frac{A}{2} \begin{pmatrix} & -i(-4-1) \\ i(4+1) & \end{pmatrix}$$

and again

$$\frac{1}{2} m_a^2 = - g^2 \, \mathrm{tr} \left([t^a, \langle \Phi \rangle] \right)^2 = \frac{25 A^2}{2} g^2$$

so all broken generators have the mass $\underline{m = 5 g A}$

SU(5) with $\langle \underset{\sim}{\Phi} \rangle = B \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$

the generators that commute with $\langle \underset{\sim}{\Phi} \rangle$ have the basis:

$$\begin{pmatrix} t^a & \\ & \end{pmatrix} \qquad \begin{pmatrix} & \\ & \tau^a \end{pmatrix} \qquad \frac{1}{\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

generate SU(3) — generate SU(2) — generates U(1)

The remaining generators have the structure

$$\frac{1}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \text{or} \quad \frac{1}{2} \begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\left[\frac{1}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \langle \underset{\sim}{\Phi} \rangle \right] = \frac{B}{2} \begin{pmatrix} & -3-2 \\ 2+3 & \end{pmatrix}$$

$$= 5 \frac{B}{2} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

again

$$\frac{1}{2} m_a^2 = -g^2 \, \mathrm{tr}\left([t^a, \langle \underset{\sim}{\Phi} \rangle] \right)^2 = \frac{25 B^2}{2} g^2$$

$$m_a = 5 g B$$

so:

$$\langle \underset{\sim}{\Phi} \rangle = A \left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \hline & & & & -4 \end{array} \right)$$

the unbroken symmetry is $SU(4) \times U(1)$ (16 generators)

the $\underline{8}$ broken generators have $m = 5gA$

$$16 + 8 = 24 \checkmark$$

$$\langle \underset{\sim}{\Phi} \rangle = B \left(\begin{array}{ccc|cc} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ \hline & & & -3 & \\ & & & & -3 \end{array} \right)$$

the unbroken symmetry is $SU(3) \times SU(2) \times U(1)$

$$8 + 3 + 1 = 12 \text{ generators}$$

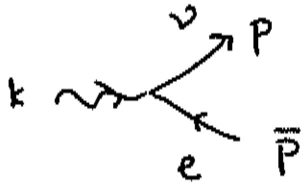
the $\underline{12}$ broken generators have $m = 5gB$

$$12 + 12 = 24 \checkmark$$

2.) a.) The decay modes of W^+ are:

$$W^+ \to e^+\nu_e \,,\; \mu^+\nu_\mu \,,\; \tau^+\nu_\tau \,,\; u\bar{d} \,,\; c\bar{s}$$

All of the partial width computations are basically the same. For $W^+ \to e^+\nu_e$



$$i\mathcal{M} = i\frac{g}{\sqrt{2}}\,\bar{u}(p)\,\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)v(\bar{p})\;\varepsilon_\mu(k)$$

$$\sum_{spin}|\mathcal{M}|^2 = \frac{g^2}{2}\,\mathrm{tr}\left[\not{p}\,\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\bar{\not{p}}\,\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\varepsilon_\mu\varepsilon_\nu^*$$

$$= \frac{g^2}{2}\,\mathrm{tr}\left[\not{p}\,\gamma^\mu\,\bar{\not{p}}\,\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right)$$

↑ by symmetry in ($\mu \leftrightarrow \nu$) the γ^5 gives zero

$$= \frac{g^2}{2}\cdot 2\left[p^\mu\bar{p}^\nu + p^\nu\bar{p}^\mu - p\cdot\bar{p}\,g^{\mu\nu}\right]\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right)$$

$$= g^2\left[p\cdot\bar{p}\,(4-2) + \frac{k\cdot p\;k\cdot\bar{p}}{m_W^2}\cdot 2 - p\bar{p}\,\frac{k^2}{m_W^2}\right]$$

Now $k = (m_W, \vec{0})$ $\quad p = (E, 0, 0, E)$ $\quad \bar{p} = (E, 0, 0, -E)$

$p\cdot\bar{p} = \dfrac{m_W^2}{2}$ $\quad k\cdot p = k\cdot\bar{p} = \dfrac{m_W^2}{2}$ $\qquad E = \dfrac{m_W}{2}$

so

$$\sum_{\text{spins}} |\mathcal{M}|^2 = g^2 \left[2 \cdot \frac{m_W^2}{2} + 2 \cdot \frac{(m_W^2/2)^2}{m_W^2} - \frac{m_W^2}{2} \right]$$

$$= g^2 m_W^2$$

then spin average is given by dividing by $\underline{3}$

$$\underline{\Gamma}(W^+ \to e^+\nu) = \frac{1}{2m_W} \cdot \frac{1}{8\pi} \cdot \frac{1}{3} g^2 m_W^2$$

$$= \frac{\alpha_w}{12} m_w$$

$$\alpha_w = \frac{g^2}{4\pi} = \frac{e^2}{4\pi \sin^2\theta_w} \cong \frac{1/129}{.23} \cong \frac{1}{30}$$

The computation for the other leptons is identical.
For the quarks, multiply by the color factor 3 and
the QCD correction $\left(1 + \frac{\alpha_s(m_W)}{\pi}\right) \cong 1 + \underline{4\%}$

$$\Gamma(W^+ \to e^+\nu_e) = \Gamma(W^+ \to \mu^+\nu_\mu) = \Gamma(W^+ \to \tau^+\nu_\tau) = \frac{\alpha_w}{12} m_W$$

$$\underline{\Gamma}(W^+ \to u\bar{d}) = \underline{\Gamma}(W^+ \to c\bar{s}) = \frac{\alpha_w}{12} m_W \cdot 3 \cdot \left(1 + \frac{\alpha_s}{\pi}\right)$$

numerically

$$\Gamma(W^+ \to e^+ \nu_e) = 0.22 \text{ GeV}$$

$$\Gamma(W^+ \to u\bar{d}) = 0.69 \text{ GeV}$$

$$\Gamma(W^+) = 2.03 \text{ GeV}$$

branching ratios:

$e^+\nu$, $\mu^+\nu$, $\tau^+\nu$	11% each	$u\bar{d}$, $c\bar{s}$	34% each

Repeat this calculation for the Z^0

for a chiral R or L-handed flavor

$$i\mathcal{M} = i \frac{e}{\cos\theta_w \sin\theta_w} \bar{u}(p)\gamma^\mu\left(\frac{1\pm\gamma^5}{2}\right) v(\bar{p})\, \epsilon_\mu(k) \cdot \left(I^3 - \sin^2\theta_w Q\right)$$

$$\frac{1}{3}\sum_{spin}|\mathcal{M}|^2 = \frac{2}{3} m_Z^2 \cdot \frac{e^2}{\sin^2\theta_w \cos^2\theta_w}\left(I^3 - \sin^2\theta_w Q\right)^2$$

then

$$\Gamma(Z^\circ \to f\bar{f}) = \frac{1}{2m_Z}\,\frac{1}{8\pi}\cdot\frac{e^2}{\sin^2\theta_w\cos^2\theta_w}\,\frac{2}{3}m_Z^2\left(I^3-\sin^2\theta_w Q\right)^2$$

$$= \frac{\alpha}{6\sin^2\theta_w\cos^2\theta_w}\,m_Z\left(I^3-\sin^2\theta_w Q\right)^2$$

$$= \frac{\alpha_w}{6\cos^2\theta_w}\,m_Z\left(I^3-\sin^2\theta_w Q\right)^2$$

the unit is:

$$\frac{\alpha_w}{6\cos^2\theta_w}\,m_Z = 0.66 \text{ GeV}$$

now account the Z° decays into all chiral flavors

$(I^3-\sin^2\theta_w Q)^2 =$	$\to f_L\bar{f}_R$		$\to f_R\bar{f}_L$	
$Z\to\nu\bar{\nu}$	$(\frac{1}{2})^2$	$= 0.25$	$\times$	
$Z\to e^-e^+$	$(-\frac{1}{2}+\sin^2\theta_w)^2 =$	0.073	$(\sin^2\theta_w)^2 =$	0.053
$Z\to u\bar{u}$	$(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w)^2 =$	0.130	$(-\frac{2}{3}\sin^2\theta_w)^2 =$	0.024
$Z\to d\bar{d}$	$(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w)^2 =$	0.179	$(\frac{1}{3}\sin^2\theta_w)^2 =$	$.006$

for the quarks, multiply by

$$3\cdot\left(1+\frac{\alpha_s}{\pi}\right) = 3.12$$

so $$\Gamma(Z^0 \to f\bar{f}) = \frac{\alpha_W m_Z}{6 \cos^2\theta_W} \cdot$$

$\nu_e \ \nu_\mu \ \nu_\tau$	0.25	× 3	=	0.750
$e \ \mu \ \tau$	0.126	×3	=	0.378
$u \ c$	0.480	×2	=	0.960
$d \ s \ b$	0.578	×3	=	1.732
				3.82

$$\Gamma(Z) = 2.5 \text{ GeV}$$

the branching fractions are

$\nu\bar{\nu}$ (invisible)	20%
e^+e^-, $\mu^+\mu^-$, $\tau^+\tau^-$	3.3% each
$u\bar{u}$ $c\bar{c}$	12.5% each
$d\bar{d}$ $s\bar{s}$ $b\bar{b}$	15% each

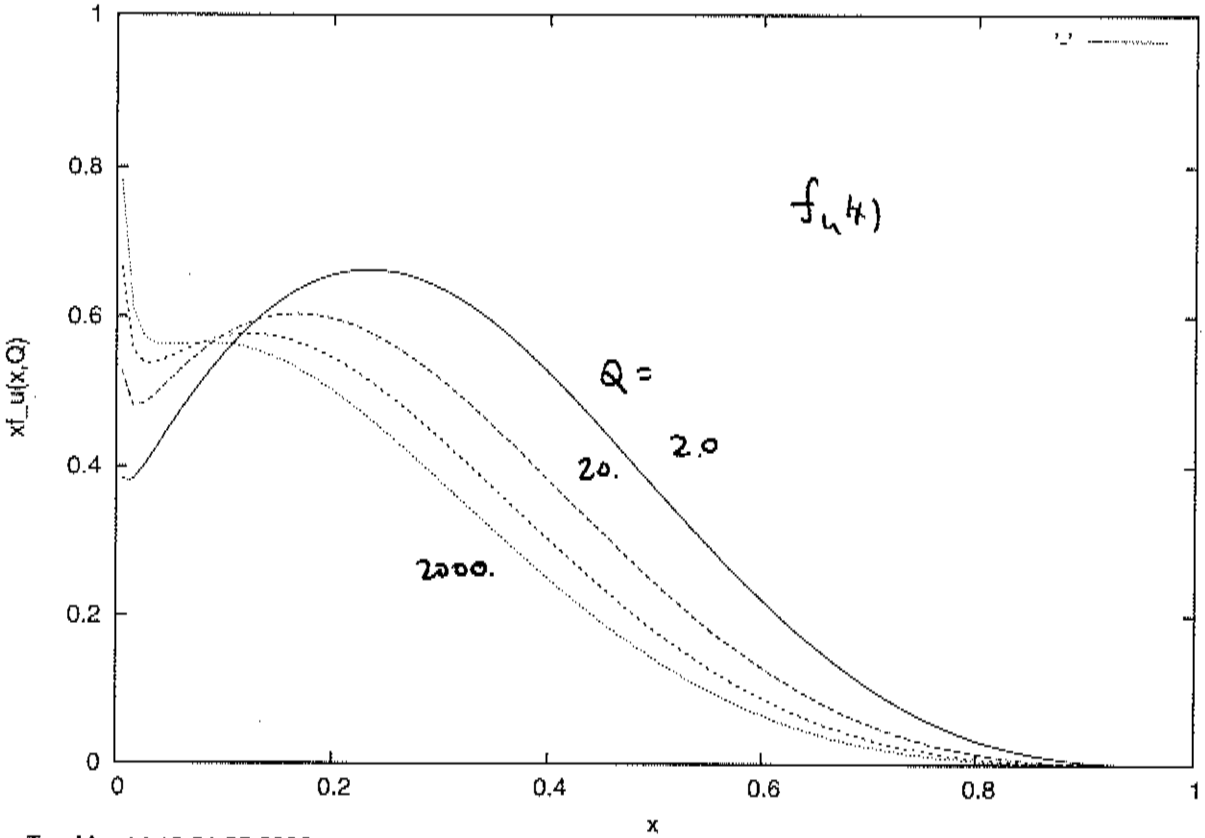
3.) To plot the evolution of d, $\bar{d}$, b quark distributions, it is only necessary to change some obvious things in the program plotxf.cpp.

In particular,

xfu → xfd or xfdbar

I have put a program plotxfmore.cpp that makes all of the required plots on the course web page.

Here are the plots →



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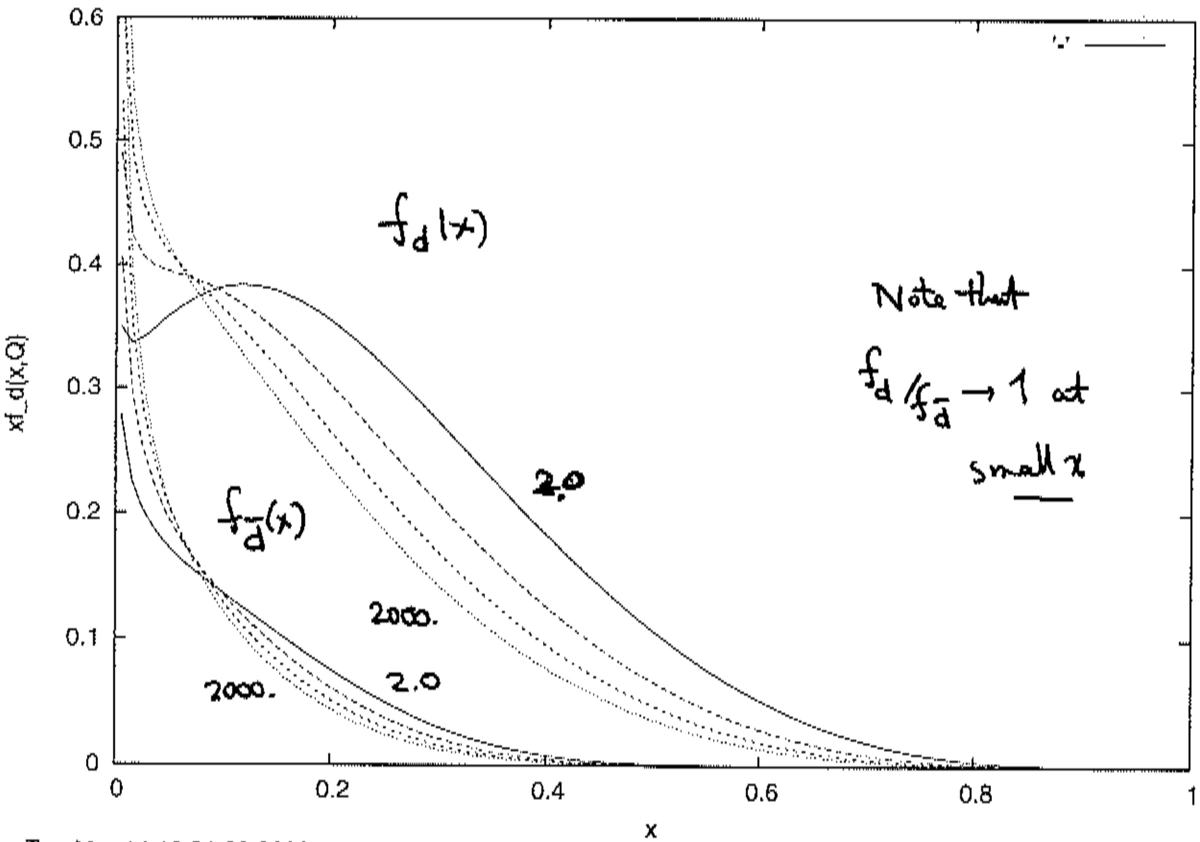

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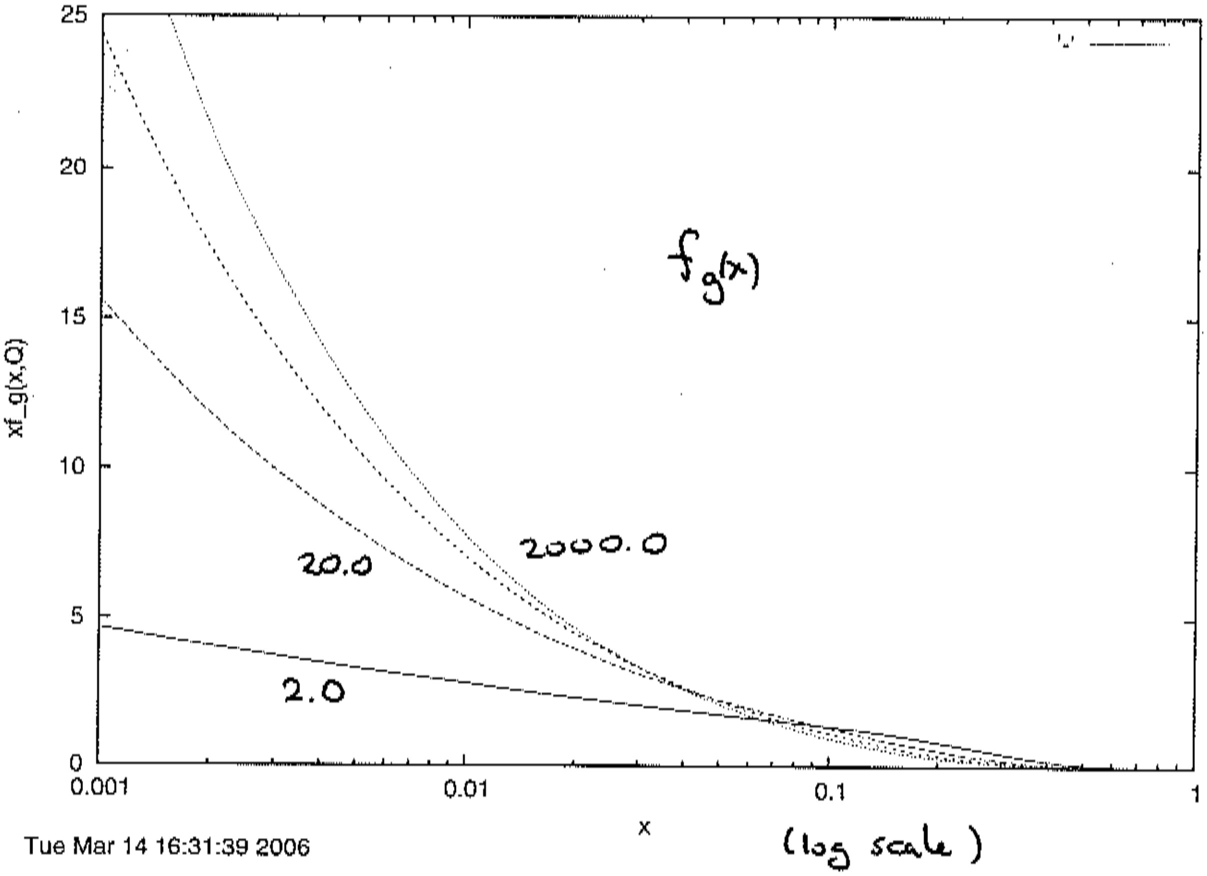
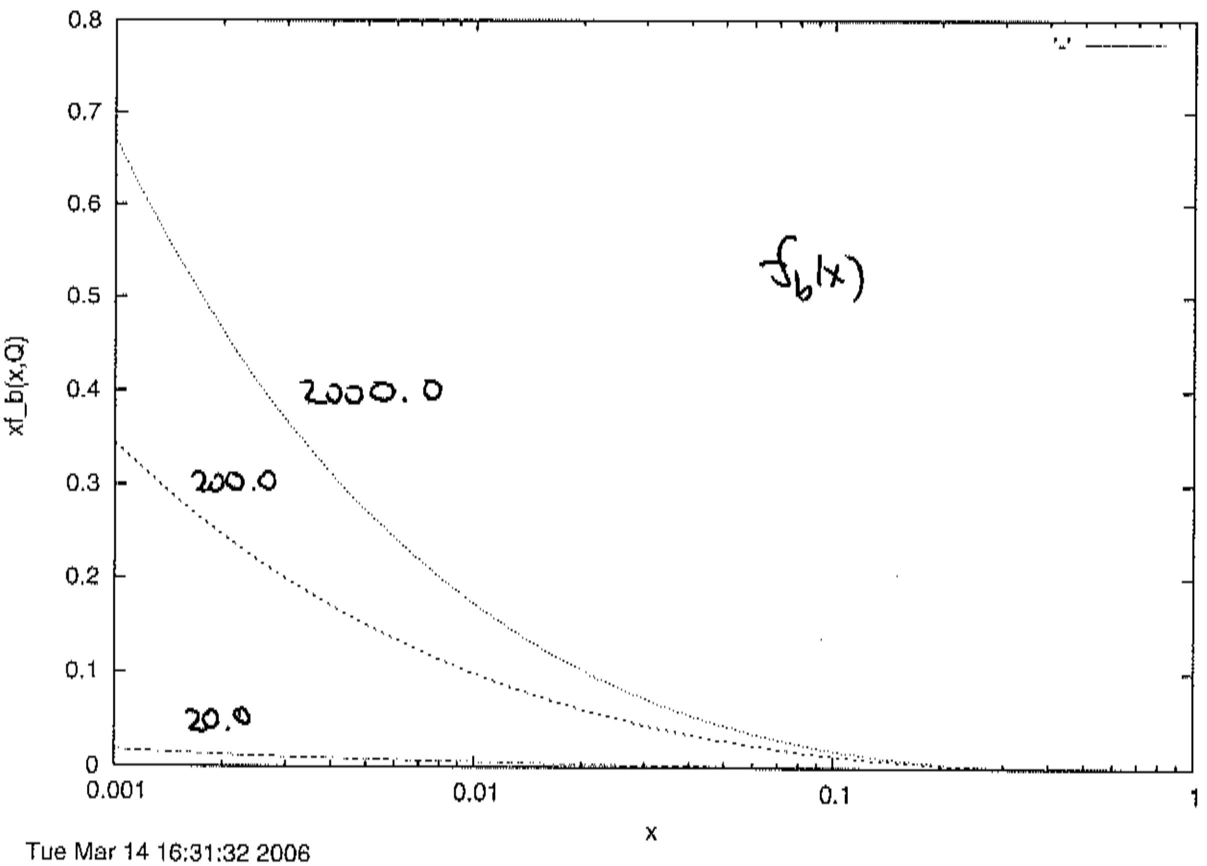


25
20
15
10
5
0
xf_g(x,Q)
0.001
0.01
0.1
1
x
f_g(x)
2000.0
20.0
2.0
(log scale)
Tue Mar 14 16:31:39 2006



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
xf_b(x,Q)
0.001
0.01
0.1
1
x
f_b(x)
2000.0
200.0
20.0
Tue Mar 14 16:31:32 2006

4.) For this problem, it is necessary to implement in ppcollider the following formulas for the total cross section:

$$\sigma = \int dx_1\, dx_2\, d\cos\Theta$$

$$\Big\{ f_g(x_1) f_g(x_2) \left[\frac{d\sigma}{d\cos\Theta}(gg \to gg) + n_f \frac{d\sigma}{d\cos\Theta}(gg \to u\bar{u}) \right]$$

$$+ 2 \sum_{f=udscb} \left[f_f(x_1) + f_{\bar{f}}(x_1) \right] f_g(x_2) \frac{d\sigma}{d\cos\Theta}(ug \to ug)$$

$$+ \sum_{f=udscb} \left(f_f(x_1) f_f(x_2) + f_{\bar{f}}(x_1) f_{\bar{f}}(x_2) \right)$$

$$\left(\frac{d\sigma}{d\cos\Theta}(u\bar{u} \to u\bar{u}) + (n_f - 1) \frac{d\sigma}{d\cos\Theta}(u\bar{u} \to d\bar{d}) + \frac{d\sigma}{d\cos\Theta}(u\bar{u} \to gg) \right)$$

$$+ 2 \sum_{f} f_f(x_1) f_{\bar{f}}(x_2) \frac{d\sigma}{d\cos\Theta}(uu \to uu)$$

$$+ \sum_{f \neq f'} \left(f_f(x_1) f_{f'}(x_2) + f_{\bar{f}}(x_1) f_{\bar{f}'}(x_2) \right) \frac{d\sigma}{d\cos\Theta}(u\bar{d} \to u\bar{d})$$

$$+ \sum_{f \neq f'} \left(f_f(x_1) f_{\bar{f}'}(x_2) + f_{\bar{f}}(x_1) f_{f'}(x_2) \right) \frac{d\sigma}{d\cos\Theta}(ud \to ud) \Big\}$$

Note that I have used

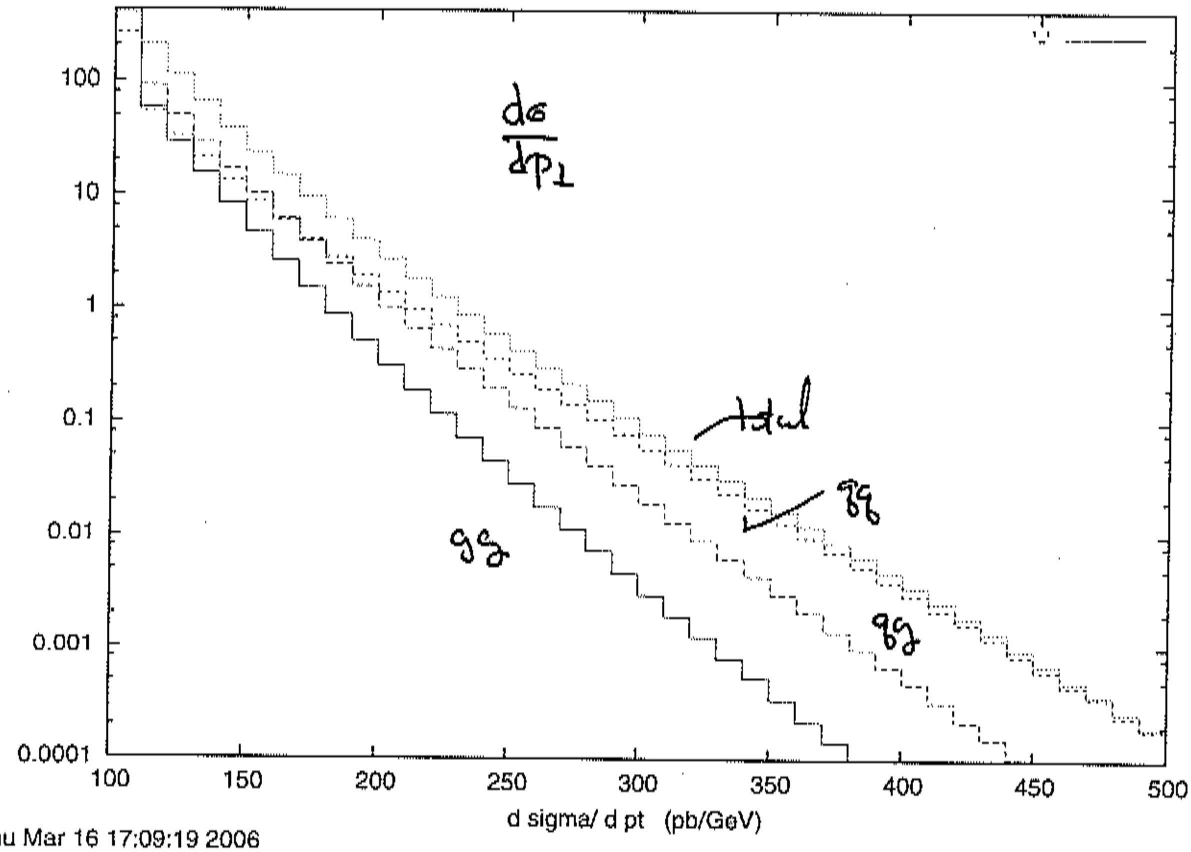
$$f_q(x) \text{ in } \bar{p} = f_{\bar{q}}(x) \text{ in } p$$

$$f_{\bar{q}}(x) \text{ in } \bar{p} = f_q(x) \text{ in } p$$

$$f_g(x) \text{ in } \bar{p} = f_g(x) \text{ in } p$$

The plots requested are shown ⟶

I have put a program ppbarcollider.cpp that makes these plots on the course Web site.



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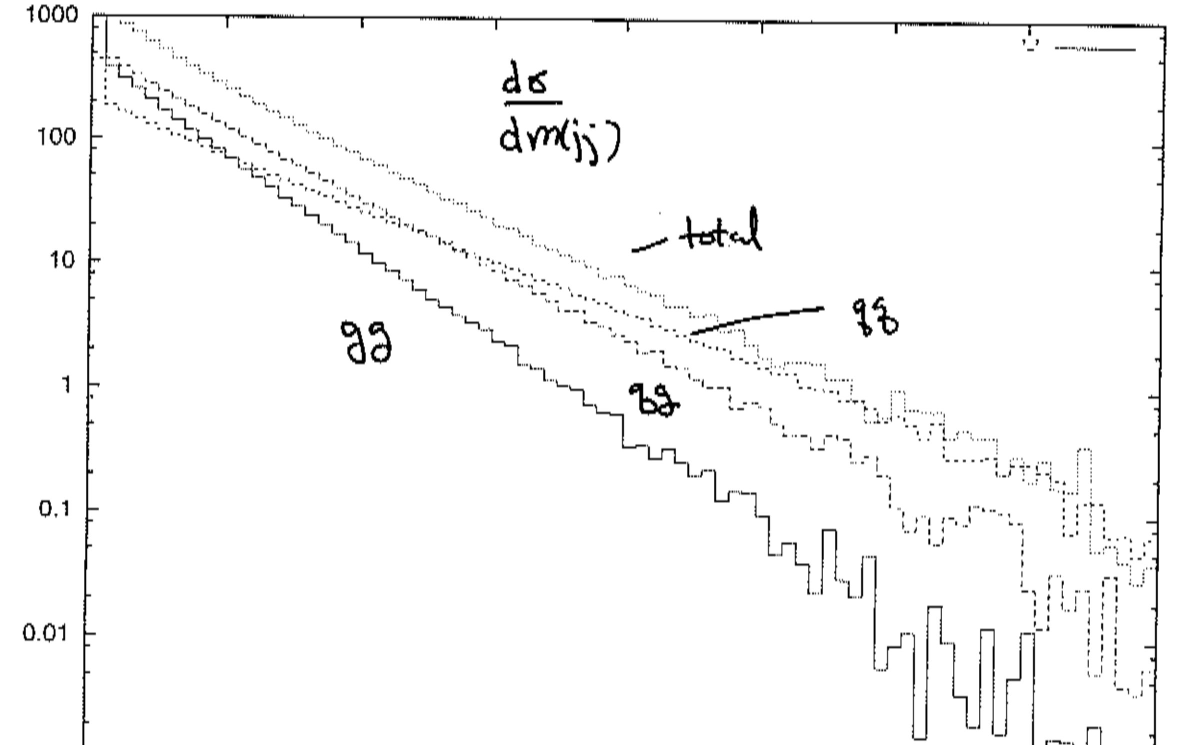

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