Physics 331 – Problem Set #5

Solutions

1.) $$\Delta \mathcal{L} = \frac{4G_F}{\sqrt{2}}\, \bar{e}_L \gamma^\nu \nu_{eL}\, \bar{\nu}_{\mu L} \gamma_\nu \mu_L$$

then

$$i\mathcal{M} = i\,\frac{4G_F}{\sqrt{2}}\, \bar{u}(p_e)\gamma^\mu P_L v(p_{\bar{\nu}})\; \bar{u}(p_\nu)\gamma_\mu P_L u(p_\mu)$$

We can do parts (a) and (b) with spin sums and projectors.

$$\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = \left(\frac{4G_F}{\sqrt{2}}\right)^2 \cdot \frac{1}{2} \cdot \mathrm{tr}\left[\not{p}_e \gamma^\mu P_L \not{p}_{\bar{\nu}} \gamma^\nu P_L\right]$$

spin avg for μ

$$\cdot\, \mathrm{tr}\left[\not{p}_\nu \gamma_\mu P_L (\not{p}_\mu + m)\gamma_\nu P_L\right]$$

$$P_L = \left(\frac{1-\gamma^5}{2}\right) \qquad P_L\gamma_\mu = \gamma_\mu P_R \qquad P_R\gamma_\nu = \gamma_\nu P_R$$

$$P_R = \left(\frac{1+\gamma^5}{2}\right) \qquad P_L^2 = P_L \qquad P_R^2 = P_R \qquad P_L \cdot P_R = 0$$

$$= 4G_F^2\; \mathrm{tr}\left[\not{p}_e \gamma^\mu \not{p}_{\bar{\nu}} \gamma^\nu P_L\right]\, \mathrm{tr}\left[\not{p}_\nu \gamma_\mu \not{p}_\mu \gamma_\nu P_L\right]$$

$$= 4G_F^2 \cdot 4 \cdot \frac{1}{2} \cdot \left[p_e^\mu p_{\bar{\nu}}^\nu + p_e^\nu p_{\bar{\nu}}^\mu - g^{\mu\nu}\, p_e \cdot p_{\bar{\nu}} + i\epsilon^{\alpha\mu\beta\nu}(p_e)_\alpha (p_{\bar{\nu}})_\beta \right]$$

$$\cdot\; 4 \cdot \frac{1}{2}\left[(p_\nu)_\mu (p_\mu)_\nu + (p_\nu)_\nu (p_\mu)_\mu - g_{\mu\nu}\, p_\nu \cdot p_\mu + i\epsilon^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} (p_\nu)_\gamma (p_\mu)_\delta \right]$$

$$= 16 G_F^2 \cdot [2\, p_e \cdot p_\nu\, p_{\bar\nu} \cdot p_\mu + 2\, p_e \cdot p_\mu\, p_{\bar\nu} \cdot p_\nu$$

$$+ (4 - 4)\, p_e \cdot p_{\bar\nu}\, p_\nu \cdot p_\mu$$

$$+ 2\, (p_e \cdot p_\nu\, p_{\bar\nu} \cdot p_\mu - p_e \cdot p_\mu\, p_{\bar\nu} \cdot p_\nu)] \leftarrow \varepsilon \cdot \varepsilon$$

$$= 64 G_F^2 \quad p_e \cdot p_\nu \;\; p_{\bar\nu} \cdot p_\mu$$

then

$$\Gamma_\mu = \frac{1}{2m_\mu} \cdot \int d\Pi_3 \quad 64 G_F^2 \; [p_e \cdot p_\nu \; p_{\bar\nu} \cdot p_\mu]$$

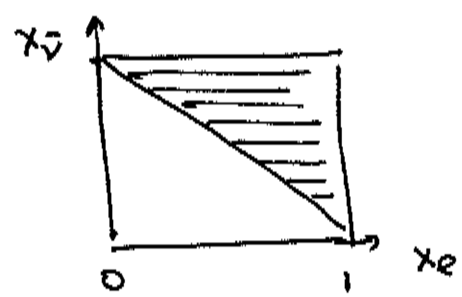
Evaluate this using tricks discussed in 330: let $p_\mu = (m, \vec{0})$

$$x_e = \frac{2p_e^0}{m} \qquad x_\nu = \frac{2p_\nu^0}{m} \qquad x_{\bar\nu} = \frac{2p_{\bar\nu}^0}{m}$$

$$x_e + x_\nu + x_{\bar\nu} = 2 \qquad\qquad 0 < x_i < 1$$

$$\int d\Pi_3 = \frac{m_\mu^2}{128\pi^3} \int dx_e\, dx_{\bar\nu}$$

over the region:



$$p_{\bar\nu} \cdot p_\mu = m\, p^0_{\bar\nu} = \tfrac{1}{2} x_{\bar\nu} m^2$$

$$p_e \cdot p_\nu = \tfrac{1}{2}(p_e + p_\nu)^2 = \tfrac{1}{2}(p_\mu - p_{\bar\nu})^2 = \tfrac{1}{2}[m^2 - 2p_\mu \cdot p_{\bar\nu}]$$

$$= \tfrac{1}{2} m^2 (1 - x_{\bar\nu})$$

so

$$I_\mu = \frac{1}{2m_\mu} \frac{m_\mu^2}{128\pi^3} \int dx_e \, dx_{\bar\nu} \quad 64 G_F^2 \cdot \tfrac{1}{4} \; x_{\bar\nu} (1 - x_{\bar\nu}) \cdot m_\mu^4$$

$$= \frac{G_F^2 m^5}{16\pi^3} \int dx_e \, dx_{\bar\nu} \quad x_{\bar\nu} (1 - x_{\bar\nu})$$

To get the total rate, integrate over x_e

$$\int_{1-x_{\bar\nu}}^{1} dx_e = x_{\bar\nu}$$

$$I_\mu = \frac{G_F^2 m^5}{16\pi^3} \int_0^1 dx_{\bar\nu} \; x_{\bar\nu}^2 (1 - x_{\bar\nu})$$

$$= \frac{G_F^2 m^5}{16\pi^3} \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{G_F^2 m^5}{192\pi^3}$$

now $$\tau_\mu = (2.20 \times 10^{-6}\,\text{sec}) \cdot \frac{1}{6.582 \times 10^{-25}\,\text{GeV sec}}$$

$$= (3.0 \times 10^{-19}\,\text{GeV})^{-1} = \Gamma_\mu^{-1}$$

then
$$G_F = \left(\frac{192\pi^3 \Gamma_\mu}{m^5}\right)^{1/2} = 1.16 \times 10^{-5}\ \text{GeV}^{-2}$$

b.) From the previous page:

$$\frac{1}{\Gamma_\mu}\frac{d^2\Gamma_\mu}{dx_e\,dx_{\bar\nu}} = 12 \cdot\ x_{\bar\nu}(1-x_{\bar\nu})$$

Integrate over $x_{\bar\nu}$:

$$\int_{1-x_e}^{1} dx_{\bar\nu}\ x_{\bar\nu}(1-x_{\bar\nu}) = \tfrac{1}{2}\left[1-(1-x_e)^2\right] - \tfrac{1}{3}\left[1-(1-x_e)^3\right]$$

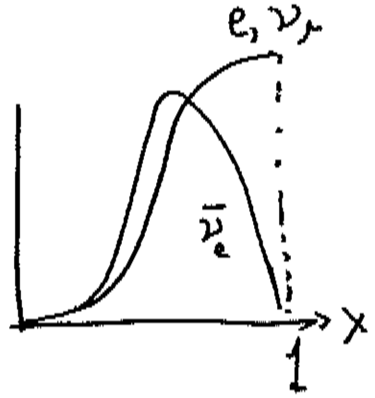
$$= x_e - \tfrac{1}{2}x_e^2 - x_e + x_e^2 - \tfrac{1}{3}x_e^3$$

$$= \tfrac{1}{2}x_e^2 - \tfrac{1}{3}x_e^3$$

$$= \tfrac{1}{2}x_e^2\left(1 - \tfrac{2}{3}x_e\right)$$

so
$$\frac{1}{\Gamma_\mu}\frac{d\Gamma_\mu}{dx_e} = 6\,x_e^2\left(1-\tfrac{2}{3}x_e\right)$$

$$\frac{1}{\Gamma_\mu}\frac{d\Gamma_\mu}{dx_{\bar\nu}} = 12\,x_{\bar\nu}^2(1-x_{\bar\nu})$$



Of course, only the e^- energy distribution is directly observable. The e^- energy distribution is in excellent agreement with experiment!

c.) Redo the calculation on p. 1 keeping the muon spinor $u(p_\mu)$ arbitrary:

It makes things easier to go back to

$$i\mathcal{M} = i\,\frac{4G_F}{\sqrt{2}}\;\bar{u}(p_e)_L\,\gamma^\mu\,v_L(p_{\bar{\nu}})\;\bar{u}_L(p_\nu)\,\gamma_\mu\,u_L(p_\mu)$$

and perform a Fierz transformation (Peskin + Schroeder (3.79))

$$= -i\,\frac{4G_F}{\sqrt{2}}\;\bar{u}_L(p_\nu)\,\gamma^\mu\,v_L(p_{\bar{\nu}})\;\bar{u}_L(p_e)\,\gamma_\mu\,u_L(p_\mu)$$

[this is why we found the same energy distribution for e and ν_μ.]

$\sum_{\text{spin}} |\mathcal{M}|^2$ contains:

$$\sum_{\text{spin}} \mathrm{tr}\;[\bar{u}_L(p_\nu)\gamma^\alpha v_L(p_{\bar{\nu}})][\bar{v}_L(p_{\bar{\nu}})\gamma^\beta u_L(p_\nu)]$$

$$= 2\,[p_\nu^\alpha p_{\bar{\nu}}^\beta + p_\nu^\beta p_{\bar{\nu}}^\alpha - g^{\alpha\beta}\,p_\nu\cdot p_{\bar{\nu}} + i\varepsilon^{\alpha\gamma\beta\delta}\,p_{\nu\gamma}\,p_{\bar{\nu}\delta}]$$

Now write the 3-body phase space as.

$$\int d\Pi_3 = \int \frac{d^3p_e}{(2\pi)^3}\frac{1}{2p_e}\;\underbrace{\int\frac{d^3p_\nu}{(2\pi)^3}\frac{1}{2p_\nu}\int\frac{d^3p_{\bar{\nu}}}{(2\pi)^3}\frac{1}{2p_{\bar{\nu}}}(2\pi)^4\delta(p_\nu+p_{\bar{\nu}}-(p_\mu-p_e))}$$

and integrate over the 2-body phase space of ν and $\bar{\nu}$.

To do this you can use the identity in the problem set:

for $k+q = P$ k, q massless

$$\int d\Pi_2 \; k^\alpha q^\beta = \frac{1}{96\pi}\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right]$$

Here is a proof! Go to a frame where $P = (M, \vec{0})$.

In the frame $k = (M/2, \frac{M}{2}\hat{n})$ $q = (M/2, -\frac{M}{2}\hat{n})$

$$\int d\Pi_2 = \frac{1}{8\pi}\int \frac{d\Omega_{\hat{n}}}{4\pi}$$

$$\int d\Pi_2 \; k^\alpha q^\beta = \frac{1}{8\pi}\int\frac{d\Omega_{\hat{n}}}{4\pi} \begin{array}{c|cc} & \beta=0 & j \\ \hline \alpha=0 & & \\ i & & \end{array}\left[\begin{array}{c|c} (M/2)^2 & -(M/2)^2\hat{n}^j \\ \hline (M/2)^2\hat{n}^i & -(M/2)^2\hat{n}^i\hat{n}^j \end{array}\right]$$

$$= \frac{1}{8\pi}\cdot\left(\frac{M}{2}\right)^2\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\frac{1}{3}\delta^{ij}\end{array}\right]$$

$$= \frac{1}{96\pi} M^2\left(\left[\begin{array}{c|c} 2 & 0 \\ \hline 0 & 0\end{array}\right] + \left[\begin{array}{c|c} 1 & \\ \hline & -\delta^{ij}\end{array}\right]\right)$$

$$= \frac{1}{96\pi}\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right]$$

If the identity is true in this frame, it is true in any frame.

then

$$\int d\Pi_2 \quad 2\left[p_\nu^\alpha p_{\bar\nu}^\beta + p_\nu^\beta p_{\bar\nu}^\alpha - g^{\alpha\beta} p_\nu \cdot p_{\bar\nu} + i\varepsilon^{\alpha\gamma\beta\delta} p_{\nu\gamma} p_{\bar\nu\delta}\right]$$

$$= \frac{1}{96\pi} \cdot 2 \cdot \Big[\left[2P^\alpha P^\beta + g^{\alpha\beta} P^2\right]\cdot 2 - g^{\alpha\beta}\left[2+4\right]P^2$$
$$+ i \cdot 0 \Big]$$

$$= \frac{1}{48\pi} \cdot \left[4P^\alpha P^\beta - 4 g^{\alpha\beta} P^2\right] \quad \text{with} \quad P = (p_\mu - p_e)$$

then

$$\int d\Pi_2 \sum_{\substack{\text{spins of}\\ e\,\nu\,\bar\nu}} |\mathcal{M}|^2 = \frac{1}{48\pi} \cdot 4\left[P^\alpha P^\beta - g^{\alpha\beta} P^2\right] \cdot \left(\frac{4G_F}{\sqrt{2}}\right)^2$$

$$\cdot \; \bar u(p_\mu)\, \gamma_\alpha \not{p}_e \gamma_\beta\, P_L\, u(p_\mu)$$

$$= \frac{8G_F^2}{12\pi}\, \bar u(p_\mu)\left[\not{P} \not{p}_e \not{P} - \gamma^\alpha \not{p}_e \gamma_\alpha P^2\right] P_L\, u(p_\mu)$$

$$\not{P}\not{p}_e\not{P} = 2P\cdot p_e \not{P} - \not{p}_e P^2$$

$$-\gamma^\alpha \not{p}_e \gamma_\alpha P^2 = 2\not{p}_e P^2$$

$$= \frac{2G_F^2}{3\pi}\, \bar u(p_\mu)\left[2(\not{p}_\mu - \not{p}_e)(p_e\cdot P) + \not{p}_e P^2\right] P_L\, u(p_\mu)$$

$$P^2 = (p_\mu - p_e)^2 = m^2 - 2p_\mu\cdot p_e = m^2(1-x_e)$$

$$2p_e\cdot P = 2p_e\cdot p_\mu = m^2 x_e$$

$$\bar u(p)\not{p}_\mu = \bar u(p_\mu)\cdot m$$

so

$$\int d\Pi_2 \sum_{spin} |M|^2 = \frac{2G_F^2}{3\pi} \bar{u}(p_\mu) \left[m^3 x_e + m^2 \not{p}_e (1-2x_e) \right] P_L u(p_\mu)$$

for a muon at rest $\quad u(p_\mu) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \bar{u}(p_\mu) = \sqrt{m} (\xi^\dagger, \xi^\dagger)$

$$P_L u(p_\mu) = \sqrt{m} \begin{pmatrix} \xi \\ 0 \end{pmatrix}$$

$$\doteq \frac{1}{3\pi} m^2 \cdot m \quad \xi^\dagger \left[x_e + \frac{\bar{\sigma} \cdot p_e}{m} (1-2x_e) \right] \xi$$

now let $\quad p_e = \left(\frac{m x_e}{2}, \frac{m x_e}{2} \sin\theta, 0, \frac{m x_e}{2} \cos\theta \right)$

$$\bar{\sigma} \cdot p_e = \frac{m x_e}{2} \begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & (1-\cos\theta) \end{pmatrix}$$

for $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^\dagger \bar{\sigma} \cdot p_e \, \xi = \frac{m x_e}{2} (1+\cos\theta)$

$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \xi^\dagger \bar{\sigma} \cdot p_e \, \xi = \frac{m x_e}{2} (1-\cos\theta)$

so

$$= \frac{2G_F^2}{3\pi} m^4 x_e \left[1 + \frac{1}{2} (1-2x_e)(1 \pm \cos\theta) \right]$$

now we can construct the total width:

$$\Gamma_\mu = \frac{1}{2m}\int d\Pi_3 \sum_{e\nu\bar{\nu}\ \text{spins}} |\mathcal{M}|^2$$

$$= \frac{1}{2m}\int \frac{d^3p_e}{(2\pi)^3}\frac{1}{2p_e}\cdot\frac{2G_F^2 m^4 x_e}{3\pi}\left[1+\frac{(1-2x_e)}{2}(1\pm\cos\theta)\right]$$

$$= \frac{1}{2m}\int \frac{dp_e\, p_e}{16\pi^3}\; 2\pi\int d\cos\theta\; \frac{2G_F^2 m^4 x_e}{2\cdot 3\pi}\left[(3-2x_e)\pm(1-2x_e)\cos\theta\right]$$

$$= \int dx_e\, x_e\left(\frac{m}{2}\right)^2\cdot\frac{1}{2m}\cdot\frac{2\pi}{16\pi^3}\int d\cos\theta\;\frac{G_F^2 m^4 x_e}{3\pi}\left[3-2x_e\pm\cdots\right]$$

$$= \frac{G_F^2 m^5}{192\pi^3}\int_0^1 dx_e\int_{-1}^{1} d\cos\theta\;\; x_e^2\left[3-2x_e \pm (1-2x_e)\cos\theta\right]$$

Integrating over $\cos\theta$ we get

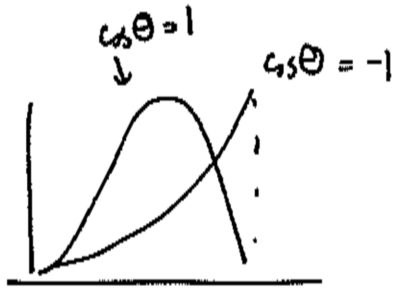
$$\Gamma_\mu = \frac{G_F^2 m^5}{192\pi^3}\cdot\int_0^1 dx_e \quad 6\,x_e^2\left(1-\tfrac{2}{3}x_e\right)$$

as required. Varying $\cos\theta$:

$$\frac{d\Gamma}{dx_e\, d\cos\theta}\sim x_e^2\left[(3-2x_e)+(1-2x_e)\cos\theta\right]$$

$\cos\theta = 1 \qquad \sim \quad 4(1-x_e)x_e^2$

$\cos\theta = -1 \qquad \sim \quad 2x_e^2$

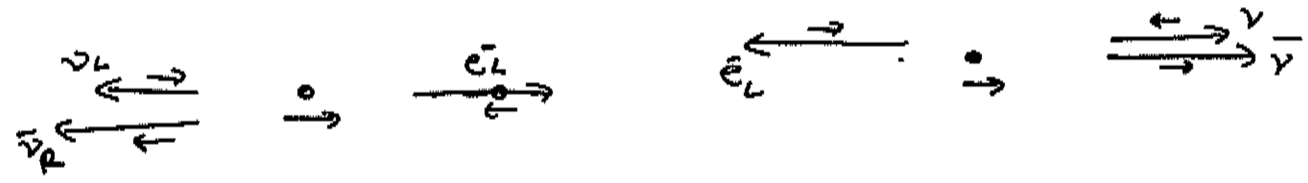


What is going on?

Consider a μ^- with spin $\parallel \hat{3}$ decays to an e^-_L, a ν_L and a $(\bar{\nu})_R$

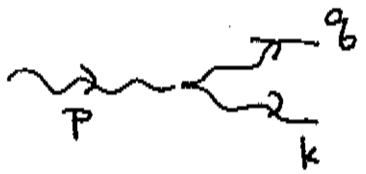
If $x_e = 1$, the momentum of the electron is maximal!

$\cos\Theta = 1$ $\qquad\qquad\qquad$ $\cos\Theta = -1$



forbidden by angular momentum $\qquad\qquad$ allowed.

The vanishing of $\dfrac{d\Gamma}{dx_e \, d\cos\Theta}$ at x_e in the direction of the muon spin has been verified in an elegant Berkeley-TRIUMF experiment.

2.) from the vertex 

we can derive the gluon splitting funct.

let $p = (p, 0, 0, p)$ $\qquad q = (zp,\ p_\perp,\ 0,\ zp)$

$$k = ((1-z)p,\ -p_\perp,\ 0,\ (1-z)p)$$

$$\varepsilon_{\substack{R\\L}}(p) = \tfrac{1}{\sqrt{2}}(0,\ 1, \pm i,\ 0)$$

$$\varepsilon_{\substack{R\\L}}(q) = \tfrac{1}{\sqrt{2}}\left(0, 1, \pm i, -\frac{p_\perp}{zp}\right)$$

$$\varepsilon_{\substack{R\\L}}(k) = \tfrac{1}{\sqrt{2}}\left(0,\ 1, \pm i, +\frac{p_\perp}{(1-z)p}\right)$$

$$i\mathcal{M} = g f^{abc}\Big[\varepsilon(p)\cdot\varepsilon^*(q)\,(p+q)\cdot\varepsilon^*(k) + \varepsilon^*(q)\cdot\varepsilon^*(k)\,(-q+k)\cdot\varepsilon^*(p) + \varepsilon^*(k)\cdot\varepsilon(p)\,(-k-p)\cdot\varepsilon^*(q)\Big]$$

$$= g f^{abc}\Big[\varepsilon(p)\cdot\varepsilon^*(q)\,[2p\cdot\varepsilon^*(k)] + \varepsilon^*(q)\varepsilon^*(k)\,[2\,k\cdot\varepsilon^*(p)] - \varepsilon^*(k)\cdot\varepsilon(p)\ 2p\cdot\varepsilon^*(q)\Big]$$

using $k\cdot\varepsilon^*(k) = 0$ etc.

Now evaluate this for the four cases
$g_R \to g_L g_L$
$g_R \to g_L g_R$
$g_R \to g_R g_L$
$g_R \to g_R g_R$

We need to work only to $\mathcal{O}(p_\perp^1)$

The other four cases are equal by P

$g_R \to g_L g_L$

$$\varepsilon_R(p)\cdot\varepsilon^*_L(q) = 0 \qquad \varepsilon^*_{qL}\cdot\varepsilon^*_{kL} = 0 \qquad \varepsilon^*_{kL}\cdot\varepsilon_{pR} = 0$$

so $i\mathcal{M} = 0$

$g_R \to g_L g_R$

$$\varepsilon_R(p)\cdot\varepsilon^*_{qL} = 0 \qquad \varepsilon^*_{qL}\cdot\varepsilon^*_{kR} = -1 \qquad \varepsilon^*_{kR}\cdot\varepsilon_{pR} = -1$$

$$i\mathcal{M} = g f^{abc}\left[\, 0 + (-1)\left(\frac{2p_\perp}{\sqrt{2}}\right) - (-1)\; 2p\cdot\frac{p_\perp}{\sqrt{2}\,z p}\right]$$

$$= g f^{abc}\,(-\sqrt{2}p_\perp)\left[\,1 - \frac{1}{z}\right] = g f^{abc}\,\sqrt{2}p_\perp\,\frac{(1-z)}{z}$$

$g_R \to g_R g_L$

$$\varepsilon_{pR}\cdot\varepsilon^*_{qR} = -1 \qquad \varepsilon^*_{qR}\cdot\varepsilon^*_{kL} = -1 \qquad \varepsilon^*_{kL}\cdot\varepsilon_{pR} = 0$$

$$i\mathcal{M} = g f^{abc}\left[(-1)\left(-\frac{2p}{\sqrt{2}}\cdot\frac{p_\perp}{(1-z)p}\right) + (-1)\,\frac{2p_\perp}{\sqrt{2}} - 0\right]$$

$$= g f^{abc}\,(\sqrt{2}p_\perp)\left(\frac{1}{(1-z)} - 1\right) = g f^{abc}(\sqrt{2}p_\perp)\,\frac{z}{(1-z)}$$

$g_R \to g_R g_R$ $\qquad \varepsilon_{pR}\cdot\varepsilon^*_{qR} = -1 \qquad \varepsilon^*_{qR}\cdot\varepsilon^*_{kR} = 0 \qquad \varepsilon^*_{kR}\cdot\varepsilon_{pR} = -1$

$$i\mathcal{M} = g f^{abc}\left[(-1)\left(-\frac{2p}{\sqrt{2}}\,\frac{p_\perp}{(1-z)p}\right) + 0 - (-1)\,\frac{2p}{\sqrt{2}}\cdot\frac{p_\perp}{zp}\right]$$

$$= g f^{abc}\,(\sqrt{2}p_\perp)\left(\frac{1}{(1-z)} + \frac{1}{z}\right) = g f^{abc}(\sqrt{2}p_\perp)\,\frac{1}{z(1-z)}$$

≈ all

$$\frac{1}{8}\sum_{\text{colors}} \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{8}\left(f^{abc}f^{abc}\right) \frac{1}{2}\, g^2 \cdot (2p_\perp^2) \cdot 2$$

$$\cdot \left[\left(\frac{1-z}{z}\right)^2 + \left(\frac{z}{(1-z)}\right)^2 + \frac{1}{(z(1-z))^2}\right]$$

$$= \frac{1}{8}\cdot 8 \cdot 3 \quad g^2 \cdot \quad 2\, p_\perp^2 \quad \frac{1}{z^2(1-z)^2} \left[(1-z)^4 + z^4 + 1 \right]$$

$$= \frac{2g^2 p_\perp^2}{z^2(1-z)^2} \cdot 3 \cdot \left[(1-z)^4 + z^4 + 1 \right]$$

to go from $\frac{1}{2}\Sigma|\mathcal{M}|^2$ to the splitting function, divide by $\dfrac{2g^2p_\perp^2}{z(1-z)}$

$$P_{g\leftarrow g}(z) = 3 \frac{1}{z(1-z)} \left[(1-z)^4 + z^4 + 1 \right]$$

$$= 3 \left[\frac{(1-z)^3}{z} + \frac{z^3}{(1-z)} + \frac{1}{z(1-z)} \right]$$

$$= 3 \left[\frac{1}{z} - 3 + 3z - z^2 + \frac{1}{1-z} - 3 + 3(1-z) - (1-z)^2 \right.$$

$$\left. + \frac{1}{z} + \frac{1}{1-z} \right]$$

$$= 3 \left[\frac{2}{z} + \frac{2}{1-z} + \left[-3 + 3z - z^2 - 3 + 3 - 3z - 1 + 2z - z^2 \right] \right]$$

$$= 3 \left[\frac{2}{z} + \frac{2}{1-z} + \left[-4 + 2z - 2z^2 \right] \right]$$

$$= 3 \left[\frac{2}{z} - 2 + \frac{2}{1-z} - 2 + 2z(1-z) \right]$$

so!

$$P_{g\leftarrow g}(z) = 6\left[\frac{(1-z)}{z} + \frac{z}{1-z} + z(1-z)\right]$$

Now, this analysis does not include the δ-function at $z=1$.

We should write more exactly:

$$P_{g\leftarrow g}(z) = P_{0,\,g\leftarrow g}(z) + A\,\delta(z-1) + B\,\delta(z-1)$$

where $B\,\delta(z-1)$ removes gluons to conserve longitudinal momentum when a gluon splits to $q\bar{q}$

$$\int dz \cdot n_f\left[z P_{q\leftarrow g}(z) + z P_{\bar{q}\leftarrow g}(z)\right] + \int dz\, z B\,\delta(z-1) = 0$$

[This is equivalent to the condition that 1 gluon is removed for each splitting:

$$\int dz \quad n_f\, P_{q\leftarrow g}(z) + B\,\delta(z-1) = 0,$$

since $\int dz\left[z\, P_{q\leftarrow g}(z) + z\, P_{\bar{q}\leftarrow g}\right]$

$$= \int dz\left[z\, P_{q\leftarrow g}(z) + (1-z)\, P_{\bar{q}\leftarrow g}\right]$$ since $P_{\bar{q}\leftarrow g}$ is symmetric under $z \to (1-z)$

$$= \int dz \quad P_{q\leftarrow g}(z)$$

Since this is $= \int dz\ \frac{1}{2}\left[z^2 + (1-z)^2\right] = \frac{1}{3}$, $\quad B = -\frac{n_f}{3}$

the condition for A is

$$\int dz\, z \left[P_{0,g \leftarrow g}(z) + A\delta(z-1) \right] = 0$$

Define $P_{0,g\leftarrow g}$ with $\frac{1}{(1-z)} \to \frac{1}{(1-z)_+}$, then

$$0 = \int dz\, z \left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + 6A\,\delta(z-1) \right]$$

$$= \int dz \left[(1-z) + \frac{z^2}{(1-z)_+} + z^2 - z^3 + 6A\,\delta(z-1) \right]$$

$$= \frac{1}{2} + \int_0^1 dz \left(\frac{z^2-1}{1-z} \right) + \frac{1}{3} - \frac{1}{4} + 6A$$

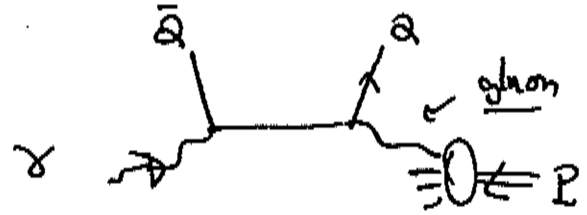
$$= \frac{1}{2} - \frac{3}{2} + \frac{1}{12} + 6A$$

$$= -\frac{11}{12} + 6A \qquad 6A = \frac{11}{12}$$

in all:

$$P_{g\leftarrow g}(z) = 6 \left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18} \right) \delta(z-1) \right]$$

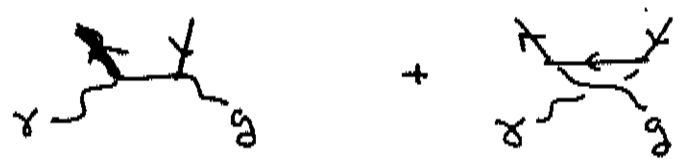
3.) For $\gamma + p \to Q\bar{Q}$, the leading parton process is:



so that

$$\sigma(\gamma + p \to Q\bar{Q} + X) = \int_0^1 dx \; f_g(x) \; \sigma(\gamma + g(xP) \to Q\bar{Q})$$

For this, we need $\quad i\mathcal{M}(\gamma + g \to Q\bar{Q})$



There is no γgg vertex, and in fact the cross section is just that of an Abelian gauge theory

In QED, the matrix element for $\gamma\gamma \to Q\bar{Q}$ is given by: (Peskin + Schroeder (5.105))

$$\frac{1}{4}\sum_{spin} |\mathcal{M}|^2 = 2e^4 \left[\frac{p_1 \cdot k_1}{p_1 \cdot k_2} + \frac{p_1 \cdot k_2}{p_1 \cdot k_1} + 2m^2 \left(\frac{1}{p_1 \cdot k_1} + \frac{1}{p_1 \cdot k_2} \right) - m^4 \left(\frac{1}{p_1 \cdot k_1} + \frac{1}{p_1 \cdot k_2} \right)^2 \right]$$

where k_1, k_2 are the photon momenta and p_1 is the Q momentum.

to go to QCD replace $e \to g$

and multiply by

$$Q_Q^2 \cdot \frac{1}{8} \operatorname{tr} t^a t^a = Q_Q^2 \cdot \frac{1}{8} \cdot C_2(r) \cdot 3 = \frac{Q_Q^2}{2}$$

so

$$\frac{1}{8}\sum_{\text{color}} \frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 (\gamma g \to Q\bar{Q}) = \frac{2e^2 g^2 Q_Q^2}{2} \quad [\text{ above }]$$

In the CM frame

$$\gamma : \quad k_1 = (E, 0\,0, E) \qquad Q : \quad p_1 = (E, ps, 0\ pc)$$
$$g : \quad k_2 = (E, 0\,0\,{-E})$$

$$p_1 \cdot k_1 = E(E-pc) \qquad p_1 \cdot k_2 = E(E+pc)$$

$$\frac{1}{8}\sum_{\text{color}} \frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = e^2 g^2 Q_Q^2 \left[\frac{E-pc}{E+pc} + \frac{E+pc}{E-pc} + \frac{2m^2}{E^2}\left(\frac{2E^2}{E^2-p^2c^2}\right) - \frac{m^4}{E^4}\left(\frac{2E^2}{E^2-p^2c^2}\right)^2 \right]$$

$$= e^2 g^2 Q_Q^2 \left[\frac{2(E^2+p^2c^2)}{E^2-p^2c^2} + 4m^2 \frac{1}{E^2-p^2c^2} - \frac{4m^4}{(E^2-p^2c^2)^2} \right]$$

$$\frac{d\sigma}{d\cos\theta_{cm}} = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{p}{E} \cdot \quad (\text{above})$$

$$\frac{d\sigma}{d\cos\theta_{cm}} = \frac{\pi\alpha\alpha_s Q_Q^2}{s} \frac{p}{E} \left[\frac{E^2+p^2c^2}{E^2-p^2c^2} + \frac{2m^2}{E^2-p^2c^2} - \frac{2m^4}{(E^2-p^2c^2)^2} \right]$$

to obtain the total cross section, integrate over $\cos\theta$. We need:

$$\int_{-1}^{1} d\cos\theta \; \frac{E^2}{E^2-p^2c^2} = \int_{-1}^{1} d\cos\theta \; \frac{E^2}{2E}\left[\frac{1}{E+p\cos\theta} + \frac{1}{E-p\cos\theta}\right]$$

$$= \frac{E^2}{2Ep}\left(\log\frac{E+p}{E-p} + \log\frac{E+p}{E-p}\right)$$

$$= \frac{E}{p}\log\frac{E+p}{E-p} = \frac{E}{p}\log\frac{(E+p)^2}{(E+p)(E-p)} = \frac{E}{p}\log\frac{(E+p)^2}{m^2}$$

$$= \frac{2E}{p}\log\frac{E+p}{m}$$

$$\int_{-1}^{1} d\cos\theta \; \frac{E^2+p^2c^2}{E^2-p^2c^2} = \int_{-1}^{1} d\cos\theta \; \frac{2E^2-(E^2-p^2c^2)}{E^2-p^2c^2}$$

$$= \frac{4E}{p}\log\frac{E+p}{m} - 2$$

$$\int_{-1}^{1} d\cos\theta \left(\frac{1}{E^2-p^2c^2}\right)^2 = \int d\cos\theta \left(\frac{1}{2E}\left[\frac{1}{(E+pc)} + \frac{1}{(E-pc)}\right]\right)^2$$

$$= \frac{1}{4E^2}\int_{-1}^{1} d\cos\theta \left[\frac{1}{(E+pc)^2} + \frac{2}{E^2-p^2c^2} + \frac{1}{(E-pc)^2}\right]$$

$$= \frac{1}{4E^2}\cdot\left\{\frac{1}{p}\left(\frac{1}{E-p} - \frac{1}{E+p}\right)\cdot 2 + \frac{4}{Ep}\log\frac{E+p}{m}\right\}$$

$$= \frac{1}{4E^2}\left\{\frac{4p}{p}\frac{1}{E^2-p^2} + \frac{4}{Ep}\log\frac{E+p}{m}\right\}$$

$$= \frac{1}{E^2 m^2} + \frac{1}{E^3 p} \lg \frac{E+p}{m}$$

$\Rightarrow$ all

$$\sigma = \frac{\pi \alpha \alpha_s Q_Q^2}{s} \frac{p}{E} \left\{ \frac{4E}{p} \lg \frac{E+p}{m} - 2 + \frac{4m^2}{Ep} \lg \frac{E+p}{m} \right.$$

$$\left. - \frac{2m^4}{E^3 p} \lg \frac{E+p}{m} - \frac{2m^4}{E^2 m^2} \right\}$$

$$= \frac{\pi \alpha \alpha_s Q_Q^2}{4E^2} \frac{p}{E} \left\{ \frac{E}{p} \lg \frac{E+p}{m} \left\{ 4 + 4\frac{m^2}{E^2} - \frac{2m^4}{E^4} \right\} \right.$$

$$\left. - 2 - \frac{2m^2}{E^2} \right\}$$

$$\sigma(\gamma g \to Q\bar{Q}) = \frac{\pi \alpha \alpha_s Q_Q^2}{4E^2} \left\{ \left(4 + 4\frac{m^2}{E^2} - 2\frac{m^4}{E^4}\right) \lg \frac{E+p}{m} \right.$$

$$\left. - 2\frac{p}{E}\left(1 + m^2/E^2\right) \right\}$$

to go ~~to~~ a general frame, write

$$4E^2 = \hat{s} \qquad 4p^2 = \hat{s} - 4m^2$$

$$\hat{s} = (k_1+k_2)^2 = 2k_1 k_2 = 2 k_1 \cdot x P = x s$$

so $\longrightarrow$

$$\sigma(\gamma p \to Q\bar{Q} + X) = \int dx \; f_g(x)$$

$$\cdot \frac{\pi \alpha \alpha_s Q_Q^2}{xs} \left\{ \left(4 + \frac{16m^2}{\hat{s}} - \frac{32m^4}{\hat{s}^2}\right) \ln\left(\frac{\hat{E}+\hat{p}}{m}\right) \right.$$

$$\left. - 2\left(1 - \frac{4m^2}{\hat{s}}\right)^{1/2}\left(1 + \frac{4m^2}{\hat{s}}\right) \right\}$$

where $\hat{s} = xs$ $\qquad \frac{\hat{E}+\hat{p}}{m} = \frac{\sqrt{\hat{s}} + \sqrt{\hat{s} - 4m^2}}{2m}$

x must be s.t. $\quad xs - 4m^2 > 0 \quad$ or $\quad x > 4m^2/s$

Now compute the cross section differential in the momentum of the Q. We have from p. 18

$$\sigma(\gamma p \to Q\bar{Q}X) = \int dx \int d\cos\theta_{cm} \; f_g(x) \cdot \frac{\pi \alpha \alpha_s Q_Q^2}{\hat{s}} \cdot \frac{P}{E}$$

$$\cdot \left[\frac{E^2 + p^2c^2}{E^2 - p^2c^2} + \frac{2m^2}{E^2 - p^2c^2} - \frac{2m^4}{(E^2 - p^2c^2)^2} \right]$$

We must change variables from $(x, \cos\theta_{cm})$ to $(p_\parallel, p_\perp)$ where $p^\mu = p_1^\mu$ is the momentum of the Q

Begin by finding x. In the CM frame of the γp collision

$$k_1 = (\mathcal{E}, 0, 0, \mathcal{E}) \qquad P = (\mathcal{E}, 0, 0, -\mathcal{E})$$

$$s = 4\mathcal{E}^2 \qquad k_2 = (x\mathcal{E}, 0, 0, -x\mathcal{E})$$

$$p_1 = (E, p_\perp, 0, p_\parallel) \qquad E^2 - p_\perp^2 - p_\parallel^2 = m^2$$

$$p_2^2 = m^2 = (k_1 + k_2 - p_1)^2 = (k_1+k_2)^2 - 2p_1(k_1+k_2) + p_1^2$$

$$m^2 = xs - 2E\cdot\mathcal{E}\cdot(1+x) + 2p_\parallel\cdot\mathcal{E}(1-x) + m^2$$

$$0 = x(s - 2\mathcal{E}E - 2\mathcal{E}p_\parallel) - 2\mathcal{E}(E - p_\parallel)$$

$$x = \frac{(E - p_\parallel)}{2\mathcal{E} - (E + p_\parallel)}$$

Next find an equation for Θ_{cm}:

$$p_\perp = p_{cm} \sin\Theta_{cm} \qquad p_{cm} = \left[\frac{xs - 4m^2}{2}\right]^{1/2}$$

$$\cos\Theta_{cm} = \left[1 - \frac{p_\perp^2}{p_{cm}^2}\right]^{1/2}$$

we need

$$\frac{\partial(x, \cos\theta_{cm})}{\partial(p_{\parallel}, p_{\perp})} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial p_{\parallel}} & \frac{\partial x}{\partial p_{\perp}} \\ \frac{\partial \cos\theta_{cm}}{\partial p_{\parallel}} & \frac{\partial \cos\theta_{cm}}{\partial p_{\perp}} \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} \\ \left.\frac{\partial \cos\theta_{cm}}{\partial x}\right|_{p_{\perp}} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial \cos\theta_{cm}}{\partial x}\right|_{p_{\perp}} \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} + \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} & \left.\frac{\partial x}{\partial p_{\perp}}\right|_{p_{\parallel}} \\ 0 & \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \end{bmatrix} \right|$$

$$= \left| \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} \cdot \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \right|$$

$$\left| \left.\frac{\partial \cos\theta_{cm}}{\partial p_{\perp}}\right|_{x} \right| = \frac{p_{\perp} / p_{cm}^2}{\left[1 - \frac{p_{\perp}^2}{p_{cm}^2}\right]^{1/2}} = \frac{p_{\perp}}{p_{\parallel}\, p_{cm}}$$

$$\left| \left.\frac{\partial x}{\partial p_{\parallel}}\right|_{p_{\perp}} \right| = \left| \frac{\left(\frac{p_{\parallel}}{E} - 1\right)}{[2\mathcal{E} - (E+p_{\parallel})]} + \frac{(E-p_{\parallel})\left(\frac{p_{\parallel}}{E}+1\right)}{[2\mathcal{E} - (E+p_{\parallel})]^2} \right|$$

$$= \left| x \cdot \left[\left(\frac{E+p_{\parallel}}{E}\right) - \frac{2\mathcal{E} - (E+p_{\parallel})}{E}\right] \frac{1}{2\mathcal{E} - E+p_{\parallel}} \right|$$

$$= \frac{2\mathcal{E} - 2(E+p_{\parallel})}{2\mathcal{E} - (E+p_{\parallel})} \frac{x}{E} = \left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}}\, x\right) \frac{x}{E}$$

so

$$dx\, d\cos\theta_{cm} = dp_{\parallel}\, dp_{\perp} \cdot \frac{p_{\perp}}{p_{\parallel}\, p_{cm}} \left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right) \frac{x}{E}$$

$$= \frac{d^3p}{E} \frac{1}{2\pi} \frac{1}{p_{\parallel}\, p_{cm}} \left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right) \cdot x$$

then

$$E \frac{d\sigma}{d^3p} (\gamma p \to Q(p)\, \bar{Q} + X)$$

$$= x f_g(x) \cdot \frac{\alpha \alpha_s Q_Q^2}{2 \hat{s}} \frac{1}{\hat{E}} \frac{\left(1 - \frac{E+p_{\parallel}}{E-p_{\parallel}} x\right)}{p_{\parallel}}$$

$$\cdot \left[\frac{\hat{E}^2 + \hat{p}^2 \cos^2\theta_{cm}}{\hat{E}^2 - \hat{p}^2 \cos^2\theta_{cm}} + \frac{2m^2}{\hat{E}^2 - \hat{p}^2 \cos^2\theta_{cm}} - \frac{2m^4}{\hat{E}^2 \hat{p}^2 \cos^2\theta_{cm}} \right]$$

where $$x = \frac{E - p_{\parallel}}{2\mathcal{E} - (E + p_{\parallel})}$$

$$\hat{s} = x s \qquad \hat{E} = \tfrac{1}{2}\sqrt{\hat{s}} \qquad \hat{p} = \left(\hat{E}^2 - m^2\right)^{1/2}$$