Physics 331 – Problem Set #4
Solutions

1.) Starting from the final formula of the previous solution set

$$\frac{d\sigma}{d\cos\theta_{cm}}(q\bar{q} \to gg) = \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{d_r}$$

$$\left[C_2(r)\left(\frac{t}{u} + \frac{u}{t}\right) - C_2(G)\,\frac{t^2+u^2}{s^2} \right]$$

set $\quad \frac{g^2}{4\pi} = \alpha_s \quad C_2(r) = \frac{4}{3} \quad d_r = 3 \quad C_2(G) = 3$

$$\frac{(C_2(r))^2}{d_r} = \frac{16}{27}$$

$$d\cos\theta_{cm} = \frac{dt}{\frac{1}{2}s} \qquad \text{since} \quad t = -\tfrac{1}{2}s(1-\cos\theta)$$

in all

$$\frac{d\sigma}{dt} = \frac{\pi}{s}\frac{1}{(s/2)}\,\alpha_s^2 \cdot \frac{16}{27}\left[\left(\frac{t}{u}+\frac{u}{t}\right) - \left(\frac{3}{4/3}\right)\frac{t^2+u^2}{s^2}\right]$$

that is:

$$\frac{d\sigma}{dt}(q\bar{q}\to gg) = \frac{32}{27}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{t}{u}+\frac{u}{t} - \frac{9}{4}\left(\frac{t^2+u^2}{s^2}\right)\right]$$

as required

for $gg\to q\bar{q}$ we simply exchange the initial and final states. The color and spin sum are:

$q\bar{q}\to gg$: $\frac{1}{4}\sum_{spin}\cdot\left(\frac{1}{3}\right)^2\sum_{color}$

$gg\to q\bar{q}$ $\frac{1}{4}\sum_{spin}\left(\frac{1}{8}\right)^2\sum_{color}$

so

$$\frac{d\sigma}{dt}(gg\to q\bar{q}) = \frac{9}{64}\times(\text{above})$$

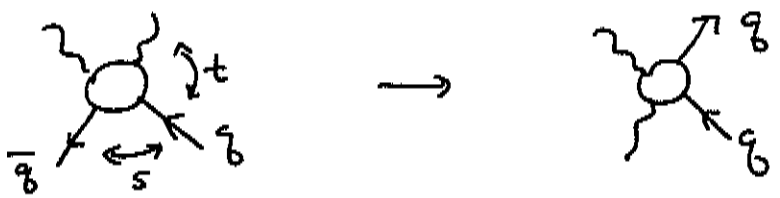
$$\frac{d\sigma}{dt}(gg\to q\bar{q}) = \frac{1}{6}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{t}{u}+\frac{u}{t} - \frac{9}{4}\,\frac{t^2+u^2}{s^2}\right]$$

for $qg\to qg$ $\frac{1}{4}\sum_{spin}\frac{1}{3\,8}\sum_{color}$

this is

ie $\frac{1}{6}\times\frac{8}{3}\to\frac{4}{9}$

and we must cross

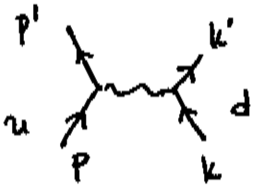


i.e. $s \leftrightarrow t \qquad u \leftrightarrow u$

There is also a (-1) when we cross a fermion into the final state. (recall Peskin + Schroeder p. 168)

$$\frac{d\sigma}{dt}(qg \to qg) = \frac{4}{9}\,\frac{\pi \alpha_s^2}{s^2}\left[-\frac{s}{u} - \frac{u}{s} + \frac{9}{4}\left(\frac{u^2+s^2}{t^2}\right)\right]$$

2.) Next, consider quark-quark scattering. First, let the quarks be of different flavors e.g. $ud \to ud$.



$$= (+ig)^2\ \bar{u}(p')\gamma^\mu u(p)\, t^a \left(\frac{-i}{t}\right) t^a\, \bar{u}(k')\gamma_\mu u(k)$$

$$\sum_{\substack{\text{spin}\\ \text{color}}} |\mathcal{M}|^2 = \frac{g^4}{t^2}\ \mathrm{tr}[\not{p}'\gamma^\mu \not{p}\gamma^\nu]\ \mathrm{tr}[\not{k}'\gamma_\mu \not{k}\gamma_\nu]\ \mathrm{tr}\, t^a t^b\ \mathrm{tr}\, t^a t^b$$

$$= \frac{g^4}{t^2}\cdot 4\,(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}\, p\cdot p')\cdot 4\,(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}\, k\cdot k')\cdot (C(r))^2 \delta^{aa}$$

$$= \frac{g^4}{t^2}\cdot 16\cdot 2\cdot (p'\cdot k'\, p\cdot k + p'\cdot k\ p\cdot k')\ d_G (C(r))^2$$

$$= 16\,\frac{g^4}{t^2}\cdot\frac{1}{2}(s^2+u^2)\cdot 8\cdot\left(\frac{1}{2}\right)^2$$

$$= \frac{16\,g^4}{t^2}(s^2+u^2)$$

To average over initial spins & colors, multiply this by $\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$

$$\frac{1}{4}\cdot\frac{1}{9}\sum_{\substack{\text{spin}\\ \text{color}}}|\mathcal{M}|^2 = \frac{4}{9}\,g^4\left(\frac{s^2+u^2}{t^2}\right)$$

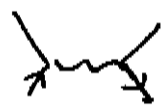
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{4}{9}\,g^4\,\frac{s^2+u^2}{t^2} = \frac{4}{9}\,\frac{1}{2s}\,\pi\alpha_s^2\left(\frac{s^2+u^2}{t^2}\right)$$

$$dt = \tfrac{1}{2}s\,d\cos\theta$$

$$\frac{d\sigma}{dt}(ud\to ud) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{s^2+u^2}{t^2}\right)$$

To compute $u\bar{d}\to u\bar{d}$ cross $s\leftrightarrow u$ $t\leftrightarrow t$

$$\frac{d\sigma}{dt}(u\bar{d}\to u\bar{d}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{s^2+u^2}{t^2}\right)$$



To compute $u\bar{u}\to d\bar{d}$ cross $s\leftrightarrow t$ $u\leftrightarrow u$

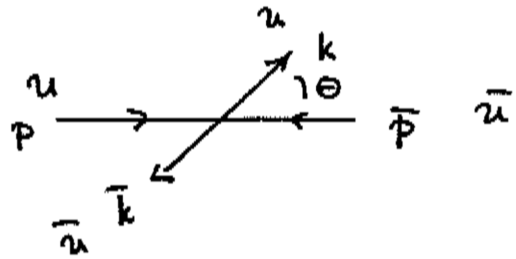
$$\frac{d\sigma}{dt}(u\bar{u}\to d\bar{d}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left(\frac{t^2+u^2}{s^2}\right)$$

The spin/color averaging factors are the same for all three cases.

Finally we need $u\bar{u} \to u\bar{u}$ 

In Peskin + Schroeder, this is done by some tricks. But it is not hard to do the calculation by straightforward evaluation of amplitudes.

Use the kinematics:



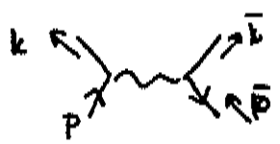
$$u_R(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_R(\bar{p}) = \sqrt{2E}\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$$

$$u_L(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v_L(\bar{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

$$u_R(k) = \sqrt{2E}\begin{pmatrix}0\\0\\c_2\\s_2\end{pmatrix} \qquad v_R(\bar{k}) = \sqrt{2E}\begin{pmatrix}-c_2\\-s_2\\0\\0\end{pmatrix}$$

$$u_L(k) = \sqrt{2E}\begin{pmatrix}-s_2\\c_2\\0\\0\end{pmatrix} \qquad v_L(\bar{k}) = \sqrt{2E}\begin{pmatrix}0\\0\\s_2\\-c_2\end{pmatrix}$$

$$c_2 = \cos\theta/2 \qquad s_2 = \sin\theta/2$$



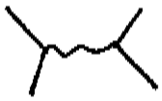
$$= (ig)^2\ \bar{u}(k)\gamma^\mu u(p)\ t^a \left(\frac{-i}{t}\right) t^a\ \bar{v}(\bar{p})\gamma_\mu v(\bar{k})$$

$$= (ig)^2\ \bar{u}(k)\gamma^\mu v(\bar{k})\ t^a \left(\frac{-i}{s}\right) t^a\ \bar{v}(\bar{p})\gamma_\mu u(p)$$

$$\times (-1)$$

from changing the order of fermions relative to the 1st diagram

By helicity conservation

 is nonzero only for $u_R \bar{u}_R \to u_R \bar{u}_R$

$u_R \bar{u}_L \to u_R \bar{u}_L$

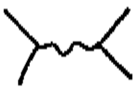
and $R \leftrightarrow L$

 is nonzero only for $u_R \bar{u}_L \to u_R \bar{u}_L$

$u_R \bar{u}_L \to u_L \bar{u}_R$

and $R \leftrightarrow L$

Evaluate this 4 amplitudes. The $R \leftrightarrow L$ amplitudes are equal by CP.

 $u_R \bar{u}_R \longrightarrow u_R \bar{u}_R$

$$i\mathcal{M} = \frac{ig^2}{t} (2E)^2 \, (c_2 \, s_2)\, \sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (-1 \; 0)\, \bar{\sigma}_\mu \begin{pmatrix} -c_2 \\ -s_2 \end{pmatrix} \quad t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 \, (c_2, s_2, is_2, c_2)^\mu g_{\mu\nu} (c_2, -s_2, is_2, -c_2)^\nu \, t^a \otimes t^a$$

$$= \frac{ig^2}{t} 4E^2 \cdot (c_2^2 + s_2^2 + s_2^2 + c_2^2) \; t^a \otimes t^a$$

$$= 2ig^2 \, \frac{s}{t} \quad t^a \otimes t^a$$

$u_R \bar{u}_L \to u_R \bar{u}_L$

$$i\mathcal{M} = \frac{ig^2}{t}(2E)^2 \,(c_2\, s_2)\,\sigma^\mu \begin{pmatrix}1\\0\end{pmatrix} \cdot (0\ -1)\,\sigma_\mu \begin{pmatrix}s_2\\-c_2\end{pmatrix} \quad t^a \otimes t^a$$

$$= \frac{ig^2}{t}\, 4E^2 \,(c_2, s_2, i s_2, c_2)^\mu g_{\mu\nu} (c_2, -s_2, -i s_2, -c_2)^\nu\, t^a \otimes t^a$$

$$= \frac{ig^2}{t}\, 4E^2 \left(c_2^2 + s_2^2 - s_2^2 + c_2^2\right) t^a \otimes t^a$$

$$= \frac{ig^2}{t}\, 4E^2 \,(2c_2^2)\, t^a \otimes t^a$$

$2\cos^2\theta/2 = 1+\cos\theta$

$u = -\frac{1}{2}s(1+\cos\theta)$

$$= -2i\, g^2 \,\frac{u}{t} \quad t^a \otimes t^a$$

$u_R \bar{u}_L \to u_R \bar{u}_L$

$$i\mathcal{M} = -i\frac{g^2}{s}(2E)^2 \,(c_2\, s_2)\,\sigma^\mu \begin{pmatrix}s_2\\-c_2\end{pmatrix} g_{\mu\nu}\, (0\ -1)\,\sigma^\nu \begin{pmatrix}1\\0\end{pmatrix} \quad t^a \otimes t^a$$

$$= -i\frac{g^2}{s}\, 4E^2 \left(0, (-c_2^2+s_2^2), i(c_2^2+s_2^2), 2c_2 s_2\right)^\mu$$

$$\cdot\, g_{\mu\nu} (0, -1, -i, 0)^\nu \quad t^a \otimes t^a$$

$$= -i\frac{g^2}{s}\, 4E^2 \cdot \left[(-c_2^2+s_2^2) - (c_2^2+s_2^2)\right] t^a \otimes t^a$$

$$= -i\frac{g^2}{s} \cdot 2\left(-2E^2(1+\cos\theta)\right) \quad t^a \otimes t^a$$

$$= -2i\,\frac{g^2}{s}\, u \cdot \quad t^a \otimes t^a$$

$u_R \bar{u}_L \to u_L \bar{u}_R$

$$i\mathcal{M} = -i\frac{g^2}{s}(2E)^2(-s_2\ c_2)\ \bar{\sigma}^\mu \begin{pmatrix}-c_2\\-s_2\end{pmatrix}\ g_{\mu\nu}\ (0\ -1)\sigma^\nu\begin{pmatrix}1\\0\end{pmatrix}\ \ t^a\otimes t^a$$

$$= -i\frac{g^2}{s}\ 4E^2\ \left(0,\ -s_2^2+c_2^2,\ i(s_2^2+c_2^2),\ -2c_2s_2\right)^\mu$$

$$\cdot\ g_{\mu\nu}\ (0\ -1,\ -i,\ 0)^\nu\ \ t^a\otimes t^a$$

$$= -i\frac{g^2}{s}\ 4E^2\ \left[\ (c_2^2-s_2^2) - (c_2^2+s_2^2)\right]\ \ t^a\otimes t^a$$

$$= -2i\frac{g^2}{s}\ (-2E^2)\ (1-\cos\theta)\ \ t^a\otimes t^a$$

$$= -2i\ \frac{g^2}{s}\ t\ \ t^a\otimes t^a$$

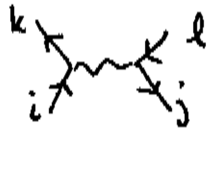
Now square the amplitudes and sum over colors. For the amplitudes with 1 diagram we can use

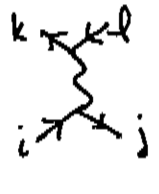
$$|t^a\otimes t^a|^2 = \mathrm{tr}\ t^a t^b\ \ \mathrm{tr}\ t^a t^b = (C(r))^2\cdot d_G = \frac{8}{2\cdot 2}$$

For $u_R\bar{u}_L \to u_R\bar{u}_L$ we must be more careful:

$$i\mathcal{M}(u_{Ri}\ \bar{u}_{Lj} \to u_{Rk}\ \bar{u}_{Ll})$$

$$= (-2ig^2)\cdot\left[\frac{u}{t}\ (t^a)_{ki}(t^a)_{jl} + \frac{u}{s}\ (t^a)_{ji}(t^a)_{kl}\right]$$





$$\sum_{\text{colors}} |\mathcal{M}|^2 = (4g^4)\Big[\left(\frac{u}{t}\right)^2 \operatorname{tr}(t^a t^b)\operatorname{tr}(t^a t^b) + \left(\frac{u}{s}\right)^2 \operatorname{tr}(t^a t^b)\operatorname{tr}(t^a t^b) + 2\,\frac{u^2}{st}\operatorname{tr}[t^a t^b t^a t^b]\Big]$$

$$= 4g^4 \left\{ \left[\left(\frac{u}{t}\right)^2 + \left(\frac{u}{s}\right)^2\right] d_G\,(C(r))^2 + 2\,\frac{u^2}{st}\cdot d_r\, C_2(r)\left(C_2(r) - \tfrac{1}{2}C_2(G)\right)\right\}$$

$$= 4g^4 \left\{ \left[\left(\frac{u}{t}\right)^2 + \left(\frac{u}{s}\right)^2\right]\cdot 2 + 2\,\frac{u^2}{st}\cdot \underbrace{3\cdot\frac{4}{3}\cdot\left(\frac{4}{3} - \frac{1}{2}\cdot 3\right)}_{(-4/6)} \right\}$$

$$= 8g^4\left[\frac{u^2}{t^2} + \frac{u^2}{s^2} + \frac{u^2}{st}\left(-\frac{2}{3}\right)\right]$$

then

$$\sum_{\text{colors}} \left\{ |\mathcal{M}|^2 (u_R\bar{u}_L \to u_R\bar{u}_L) + |\mathcal{M}|^2 (u_R\bar{u}_R \to u_R\bar{u}_R) + |\mathcal{M}|^2 (u_R\bar{u}_L \to u_L\bar{u}_R)\right\}$$

$$= 8g^4\left[\frac{u^2}{t^2} + \frac{u^2}{s^2} - \frac{2}{3}\frac{u^2}{st} + \frac{s^2}{t^2} + \frac{t^2}{s^2}\right]$$

$$\frac{1}{4}\frac{1}{9}\sum_{\text{spin, color}} |\mathcal{M}|^2 = \frac{1}{4}\cdot\frac{1}{9}\cdot 2\times(\text{above})$$

$$= \frac{4}{9}g^4\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} - \frac{2}{3}\frac{u^2}{st}\right]$$

Finally, $$\frac{d\sigma}{d\cos\theta}(u\bar{u}\to u\bar{u}) = \frac{1}{2s}\,\frac{1}{16\pi}\,\frac{4}{9}\,g^4\,[\cdots]$$

$$\frac{d\sigma}{dt}(u\bar{u}\to u\bar{u}) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{s^2+u^2}{t^2}+\frac{t^2+u^2}{s^2}-\frac{2}{3}\,\frac{u^2}{st}\right]$$

to obtain $uu\to uu$ or $\bar{u}\bar{u}\to\bar{u}\bar{u}$,

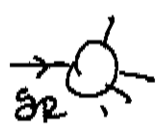
cross $\quad s\leftrightarrow u \quad t\leftrightarrow t$

$$\frac{d\sigma}{dt}(uu\to uu) = \frac{4}{9}\,\frac{\pi\alpha_s^2}{s^2}\left[\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}-\frac{2}{3}\,\frac{s^2}{ut}\right]$$

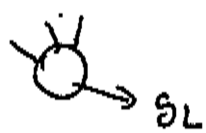
3.) Finally, we need to compute $gg\to gg$. Begin by computing the amplitudes for

$$g_R g_R \to g_L g_L$$
$$g_R g_R \to g_R g_L$$
$$g_R g_R \to g_R g_R$$

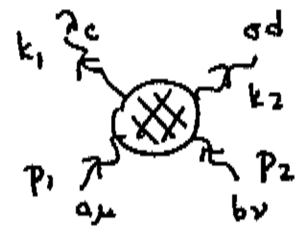
The other needed helicity amplitudes can be obtained from these by P and <u>crossing</u>

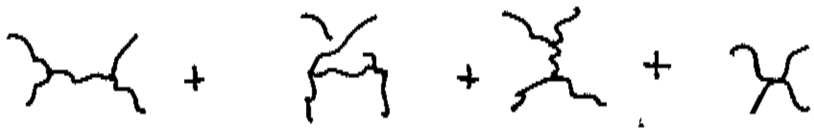


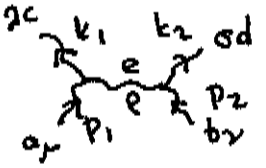
crosses to



Set up the calculation as



There are 4 diagrams: 



$$= (g f^{ace}) \left[g^{\mu\lambda}(p_1+k_1)^\rho + g^{\lambda\rho}(-k_1-(k_1-p_1))^\mu + g^{\rho\mu}(k_1-p_1-p_1)^\lambda \right]$$

$$\cdot \frac{-i}{t} (g f^{bde}) \left[g^{\nu\sigma}(p_2+k_2)^\rho + g^{\sigma\rho}(-k_2-(k_2-p_2))^\nu + g^{\rho\nu}(k_2-p_2-p_2)^\sigma \right]$$

$$\varepsilon^*_\lambda(k_1)\, \varepsilon^*_\sigma(k_2)\, \varepsilon_\mu(p_1)\, \varepsilon_\nu(p_2)$$

Using $p\cdot\varepsilon(p) = 0$:

$$= g f^{ace} \left[\varepsilon_1\cdot\varepsilon_1^* (p_1+k_1)^\rho - 2 k_1\cdot\varepsilon_1\, \varepsilon_1^{*\rho} + (-2)\, p_1\cdot\varepsilon_1^*\, \varepsilon_1^\rho \right] \frac{-i}{t}$$

$$\cdot\, g f^{bde} \left[\varepsilon_2\cdot\varepsilon_2^* (p_2+k_2)^\rho - 2k_2\cdot\varepsilon_2\, \varepsilon_2^{*\rho} - 2\, p_2\cdot\varepsilon_2^*\, \varepsilon_2^\rho \right]$$

$$= \frac{-ig^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^*\ \varepsilon_2\cdot\varepsilon_2^*\ (p_1+k_1)\cdot(p_2+k_2)$$

$$+ \varepsilon_1\cdot\varepsilon_1^* (-2)\cdot \left[\varepsilon_2\cdot k_2\ \varepsilon_2^*\cdot(p_1+k_1) + \varepsilon_2^* p_2\ \varepsilon_2\cdot(p_1+k_1) \right]$$

$$+ \varepsilon_2\cdot\varepsilon_2^* (-2) \left[\varepsilon_1\cdot k_1\ \varepsilon_1^* (p_2+k_2) + \varepsilon_1^* p_1\ \varepsilon_1\cdot(p_2+k_2) \right]$$

$$+ 4 \big[\varepsilon_1^*\cdot\varepsilon_2^*\ k_1\cdot\varepsilon_1\ k_2\cdot\varepsilon_2 + \varepsilon_1^*\cdot\varepsilon_2\ k_1\cdot\varepsilon_1\ p_2\cdot\varepsilon_2^*$$

$$+ \varepsilon_1\cdot\varepsilon_2^*\ p_1\cdot\varepsilon_1^*\ k_2\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2\ p_1\cdot\varepsilon_1^*\ p_2\cdot\varepsilon_2^* \big] \Big\}$$

$$= \begin{pmatrix} c \\ k_1 \\ \varepsilon_1^* \end{pmatrix} \leftrightarrow \begin{pmatrix} d \\ k_2 \\ \varepsilon_2^* \end{pmatrix} \text{ in the above}$$

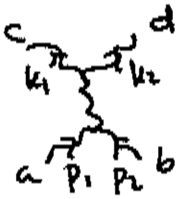
$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; (p_1+k_2)\cdot(p_2+k_1)$$

$$-2\,\varepsilon_1\varepsilon_2^* \left[\varepsilon_2\cdot k_1 \; \varepsilon_1^*(p_1+k_2) + \varepsilon_1^*\cdot p_2 \; \varepsilon_2(p_1+k_2) \right]$$

$$-2\,\varepsilon_2\cdot\varepsilon_1^* \left[\varepsilon_1\cdot k_2 \; \varepsilon_2^*(p_2+k_1) + \varepsilon_2^*\cdot p_1 \; \varepsilon_1\cdot(p_2+k_1) \right]$$

$$+ 4 \left[\varepsilon_1^*\cdot\varepsilon_2^* \; k_2\cdot\varepsilon_1 \; k_1\cdot\varepsilon_2 + \varepsilon_2^*\cdot\varepsilon_2 \; \varepsilon_1\cdot k_2 \; \varepsilon_1^*\cdot p_2 \right.$$

$$\left. + \varepsilon_1\cdot\varepsilon_1^* \; p_1\cdot\varepsilon_2^* \; k_1\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2 \; p_1\varepsilon_2^* \; p_2\cdot\varepsilon_1^* \right] \Big\}$$



$$= g f^{abe} \left[g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2+(p_1+p_2))^\mu + g^{\rho\mu}(-p_1-p_2-p_1)^\nu \right]$$

$$\cdot \frac{-i}{s} g f^{cde} \left[g^{\lambda\sigma}(-k_1+k_2)^\rho + g^{\sigma\rho}(-k_2-(k_1+k_2))^\lambda + g^{\rho\lambda}((k_1+k_2)+k_1)^\sigma \right]$$

$$\cdot\, \varepsilon_1^\mu \varepsilon_2^\nu \; \varepsilon_1^{*\lambda} \varepsilon_2^{*\sigma}$$

$$= \frac{-ig^2 f^{abe} f^{cde}}{s} \left[\varepsilon_1\varepsilon_2 (p_1-p_2)^\rho + 2\varepsilon_1\cdot p_2\, \varepsilon_2^\rho - 2\varepsilon_2\cdot p_1\, \varepsilon_1^\rho \right]$$

$$\left[-\varepsilon_1^*\varepsilon_2^*(k_1-k_2)^\rho - 2\varepsilon_1^*\cdot k_2\, \varepsilon_2^{*\rho} + 2\varepsilon_2^*\cdot k_1\, \varepsilon_1^{*\rho} \right]$$

$$= \frac{+ig^2 f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* \; (p_1-p_2)\cdot(k_1-k_2)$$

$$+2\,\varepsilon_1\varepsilon_2 \left[\varepsilon_1^* k_2 \; \varepsilon_2^*(p_1-p_2) - \varepsilon_2^* k_1 \; \varepsilon_1^*\cdot(p_1-p_2) \right]$$

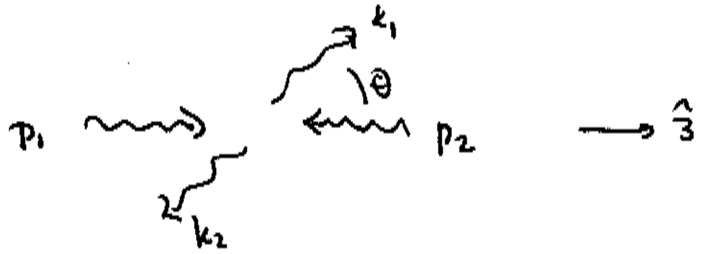
$$+2\,\varepsilon_1^*\varepsilon_2^* \left[\varepsilon_1\cdot p_2 \; \varepsilon_2(k_1-k_2) - \varepsilon_2\cdot p_1 \; \varepsilon_1\cdot(k_1-k_2) \right]$$

+ (more) $\longrightarrow$

$$+\,4\,\big[\ \varepsilon_2\cdot\varepsilon_2^*\ \ \varepsilon_1\cdot p_2\ \ \varepsilon_1^*\cdot k_2\ -\ \varepsilon_2\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_2^*\cdot k_1$$
$$-\ \varepsilon_1\cdot\varepsilon_2^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_1^*\cdot k_2\ +\ \varepsilon_1\cdot\varepsilon_1^*\ \ \varepsilon_2\cdot p_1\ \ \varepsilon_2^*\cdot k_1\big]\ \Big\}$$

$$= \ -ig^2\Big[\, f^{abe}f^{cde}\ (\varepsilon_1\cdot\varepsilon_1^*\ \varepsilon_2\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^*\ \varepsilon_2\cdot\varepsilon_1^*)$$
$$+\ f^{ace}f^{bde}\ (\varepsilon_1\cdot\varepsilon_2\ \ \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^*\ \varepsilon_2\cdot\varepsilon_1^*)$$
$$+\ f^{ade}f^{bce}\ (\varepsilon_1\cdot\varepsilon_2\ \ \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_1^*\ \varepsilon_2\cdot\varepsilon_2^*)\Big]$$

Now evaluate these with explicit ε's.



$c = \cos\theta$
$s = \sin\theta$

$p_1 = (E, 0, 0, E)$

$$\varepsilon_{1R}^{\mu} = \tfrac{1}{\sqrt{2}}\,(0, 1, i, 0)^{\mu}$$
$$\varepsilon_{1L}^{\mu} = \tfrac{1}{\sqrt{2}}\,(0, 1, -i, 0)^{\mu}$$

$p_2 = (E, 0, 0, -E)$

$$\varepsilon_{2R}^{\nu} = \tfrac{1}{\sqrt{2}}\,(0, -1, i, 0)^{\nu}$$
$$\varepsilon_{2L}^{\nu} = \tfrac{1}{\sqrt{2}}\,(0, -1, -i, 0)^{\nu}$$

$k_1 = (E, Es, 0, Ec)$

$$\varepsilon_{1R}^{*\lambda} = \tfrac{1}{\sqrt{2}}\,(0, c, -i, -s)^{\lambda}$$
$$\varepsilon_{1L}^{*\lambda} = \tfrac{1}{\sqrt{2}}\,(0, c, +i, -s)^{\lambda}$$

$k_2 = (E, -Es, 0, -Ec)$

$$\varepsilon_{2R}^{*\sigma} = \tfrac{1}{\sqrt{2}}\,(0, -c, -i, s)^{\sigma}$$
$$\varepsilon_{2L}^{*\sigma} = \tfrac{1}{\sqrt{2}}\,(0, -c, +i, s)^{\sigma}$$

It is useful to make some tables of dot products:

	ε_{1R}	ε_{1L}	ε_{2R}	ε_{2L}	ε_{1R}^*	ε_{1L}^*	ε_{2R}^*	ε_{2L}^*
ε_{1R}	0	-1	1	0	$-\left(\frac{1+c}{2}\right)$	$\frac{1-c}{2}$	$-\frac{(1-c)}{2}$	$\left(\frac{1+c}{2}\right)$
ε_{1L}	-1	0	0	1	$\frac{1-c}{2}$	$-\left(\frac{1+c}{2}\right)$	$\left(\frac{1+c}{2}\right)$	$-\frac{(1-c)}{2}$
ε_{2R}			0	-1	$-\left(\frac{1-c}{2}\right)$	$\left(\frac{1+c}{2}\right)$	$-\frac{(1+c)}{2}$	$\frac{(1-c)}{2}$
ε_{2L}			-1	0	$\left(\frac{1+c}{2}\right)$	$-\left(\frac{1-c}{2}\right)$	$\frac{(1-c)}{2}$	$-\frac{(1+c)}{2}$
ε_{1R}^*					0	-1	1	0
ε_{1L}^*					-1	0	0	1
ε_{2R}^*							0	-1
ε_{2L}^*							-1	0

	$\underbrace{\varepsilon_{1R} \quad \varepsilon_{1L}}$	$\underbrace{\varepsilon_{2R} \quad \varepsilon_{2L}}$	$\underbrace{\varepsilon_{1R}^* \quad \varepsilon_{1L}^*}$	$\underbrace{\varepsilon_{2R}^* \quad \varepsilon_{2L}^*}$
p_1	0	0	$\frac{E_s}{\sqrt{2}}$	$-\frac{E_s}{\sqrt{2}}$
p_2	0	0	$-\frac{E_s}{\sqrt{2}}$	$\frac{E_s}{\sqrt{2}}$
k_1	$-\frac{E_s}{\sqrt{2}}$	$\frac{E_s}{\sqrt{2}}$	0	0
k_2	$\frac{E_s}{\sqrt{2}}$	$-\frac{E_s}{\sqrt{2}}$	0	0

Since the dot products $p_i\cdot\varepsilon^*$ $k_i\cdot\varepsilon$ are independent of shown helicity, we might as well put in their values and simplify the diagrams:

$$= \frac{-ig^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^* \, \varepsilon_2\cdot\varepsilon_2^* \, (s-u)$$

$$- 2\, \varepsilon_1\cdot\varepsilon_1^* \left[\left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) + \left(\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) \right]$$

$$- 2\, \varepsilon_2\cdot\varepsilon_2^* \left[\left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) + \left(\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) \right]$$

$$+ 4 \Big[\varepsilon_1^*\cdot\varepsilon_2^* \left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) + \varepsilon_1^*\varepsilon_2 \left(-\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right)$$

$$+ \varepsilon_1\cdot\varepsilon_2^* \left(\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) + \varepsilon_1\cdot\varepsilon_2 \left(\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) \Big] \Big\}$$

$$= \frac{-ig^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^* \, \varepsilon_2\cdot\varepsilon_2^* \, (s-u)$$

$$- 2\,(Es)^2 \Big[\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1^*\cdot\varepsilon_2 + \varepsilon_1\cdot\varepsilon_2^*$$

$$- \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2 \Big] \Big\}$$

$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\varepsilon_2^* \, \varepsilon_2\cdot\varepsilon_1^* \, (s-t)$$

$$- 2\, \varepsilon_1\varepsilon_2^* \left[\left(+\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) + \left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) \right]$$

$$- 2\, \varepsilon_2\cdot\varepsilon_1^* \left[\left(\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) + \left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) \right]$$

$$+ 4 \Big[\varepsilon_1^*\cdot\varepsilon_2^* \left(\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) + \varepsilon_2\cdot\varepsilon_2^* \left(\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right)$$

$$+ \varepsilon_1\varepsilon_1^* \left(-\tfrac{Es}{\sqrt{2}}\right)\left(\tfrac{Es}{\sqrt{2}}\right) + \varepsilon_1\cdot\varepsilon_2 \left(-\tfrac{Es}{\sqrt{2}}\right)\left(-\tfrac{Es}{\sqrt{2}}\right) \Big]$$

$$= \frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \,(s-t) - 2\,(Es)^2 \big[\varepsilon_1\cdot\varepsilon_2^* + \varepsilon_2\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1\cdot\varepsilon_1^* - \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2 \big] \Big\}$$

$$= \frac{+ig^2 f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* \,(u-t) + 2\,\varepsilon_1\cdot\varepsilon_2\,[\,0\,] + 2\,\varepsilon_1^*\cdot\varepsilon_2^*\,[\,0\,] + 4\,[\,0\,] \Big\}$$

$$\begin{aligned} = & -ig^2 f^{abe} f^{cde} \{ \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \} \\ & -ig^2 f^{ace} f^{bde} \{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \} \\ & -ig^2 f^{ade} f^{bce} \{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\cdot\varepsilon_2^* - \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \} \end{aligned}$$

in all $\longrightarrow$

$$= \frac{-ig^2 f^{abe} f^{cde}}{s} \Big\{ \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* \; (t-u)$$

$$+ \; \varepsilon_1\varepsilon_1^* \, \varepsilon_2\varepsilon_2^* \cdot s \; - \; \varepsilon_1\varepsilon_2^* \; \varepsilon_2\varepsilon_1^* \; s \Big\}$$

$$\frac{-ig^2 f^{ace} f^{bde}}{t} \Big\{ \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \; (s-u)$$

$$+ \; \varepsilon_1\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* \cdot t \; - \; \varepsilon_1\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \cdot t$$

$$-2(Es)^2 \left[\varepsilon_1\varepsilon_1^* + \varepsilon_2\varepsilon_2^* + \varepsilon_1^*\varepsilon_2 + \varepsilon_2^*\varepsilon_1 - \varepsilon_1^*\varepsilon_2^* - \varepsilon_1\varepsilon_2 \right] \Big\}$$

$$\frac{-ig^2 f^{ade} f^{bce}}{u} \Big\{ \varepsilon_1\varepsilon_2^* \; \varepsilon_2\cdot\varepsilon_1^* \; (s-t)$$

$$+ \; \varepsilon_1\cdot\varepsilon_2 \; \varepsilon_1^*\varepsilon_2^* \cdot u \; - \; \varepsilon_1\cdot\varepsilon_1^* \; \varepsilon_2\cdot\varepsilon_2^* \cdot u$$

$$-2(Es)^2 \left[\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1^*\varepsilon_2 + \varepsilon_2^*\varepsilon_1 - \varepsilon_1^*\varepsilon_2^* - \varepsilon_1\varepsilon_2 \right] \Big\}$$

Now it is not so hard to evaluate this for all 16 helicity amplitudes. However, <u>three</u> will suffice.

for $g_R g_R \longrightarrow g_L g_L$

$$\varepsilon_1 \cdot \varepsilon_2 = \varepsilon_1^* \cdot \varepsilon_2^* = 1 \qquad \varepsilon_1 \cdot \varepsilon_1^* = \frac{1-c}{2} = \varepsilon_2 \cdot \varepsilon_2^*$$

$$\varepsilon_1 \cdot \varepsilon_2^* = \frac{1+c}{2} = \varepsilon_2 \cdot \varepsilon_1^*$$

$$\varepsilon_1 \cdot \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_1^* \varepsilon_2 + \varepsilon_2^* \varepsilon_1 - \varepsilon_1^* \cdot \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2$$

$$= (1-c) + (1+c) - 1 - 1 = 0$$

then

$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ t-u + \left(\frac{1-c}{2}\right)^2 s - \left(\frac{1+c}{2}\right)^2 s \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \left(\frac{1-c}{2}\right)^2 (s-u) + t - \left(\frac{1+c}{2}\right)^2 t \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \left(\frac{1+c}{2}\right)^2 (s-t) + u - \left(\frac{1-c}{2}\right)^2 u \right\}$$

now $\quad s = 4E^2 \quad t = -2E^2(1-c) \quad u = -2E^2(1+c)$

$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ 4E^2 c - \frac{1}{4} \cdot 4c \cdot 4E^2 \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \frac{(1-c)^2}{4} (2E^2)(3+c) - \frac{2E^2}{4}(1-c)\left[4-(1+c)^2\right] \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \frac{(1+c)^2}{4} (2E^2)(3-c) - \frac{2E^2}{4}(1+c)\left[4-(1-c)^2\right] \right\}$$

All the lines are zero !

coeff of $\frac{1}{s}$: $\quad \{ 4E^2 c - \frac{1}{4} c \, 4E^2 \} = 0$

coeff of $\frac{1}{t}$: $\quad \{ \frac{(1-c)}{4} 2E^2 \cdot (1-c)(3+c) - \frac{(1-c)\,2E^2}{4} [3 - 2c - c^2] \} = 0$

coeff of $\frac{1}{u}$: $\quad \{ \frac{(1+c)}{4} 2E^2 (1+c)(3-c) - \frac{(1+c)\,2E^2}{4} [3 + 2c - c^2] \} = 0$

so $\quad i\mathcal{M}(g_R g_R \to g_L g_L) = 0 \quad$ and also (by P)

$$i\mathcal{M}(g_L g_L \to g_R g_R) = 0$$

for $\quad \underline{g_R g_R \to} g_R g_L$

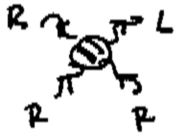
$$\varepsilon_1 \varepsilon_2 = 1 \qquad \varepsilon_1^* \cdot \varepsilon_2^* = 0 \qquad \varepsilon_1 \cdot \varepsilon_1^* = -\left(\frac{1+c}{2}\right) \qquad \varepsilon_2 \cdot \varepsilon_2^* = \frac{1-c}{2}$$

$$\varepsilon_1 \cdot \varepsilon_2^* = \frac{1+c}{2} \qquad \varepsilon_2 \cdot \varepsilon_1^* = -\frac{(1-c)}{2}$$

$$\varepsilon_1 \cdot \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_1 \cdot \varepsilon_2^* + \varepsilon_2 \cdot \varepsilon_1^* - \varepsilon_1^* \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2$$

$$= -\left(\frac{1+c}{2}\right) + \left(\frac{1-c}{2}\right) + \left(\frac{1+c}{2}\right) - \frac{(1-c)}{2} - 0 - 1 = -1$$

then



$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ 0 + \left[-\frac{(1+c)(1-c)}{4} \right] s - \left[-\frac{(1+c)(1-c)}{4} \right] s \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ -\left(\frac{1-c^2}{4}\right)(s-u) + 0 - \left[-\left(\frac{1-c^2}{4}\right) \right] t + 2E^2 s^2 \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ -\frac{(1-c^2)}{4}(s-t) + 0 - \left[-\left(\frac{1-c^2}{4}\right) \right] u + 2E^2 s^2 \right\}$$

The coefficient of $\frac{1}{s}$ is $\left[-\underset{\substack{\uparrow\\ \sin\theta}}{\frac{s^2}{4}}\cdot \underset{\substack{\uparrow\\ 4E^2}}{s} + \frac{s^2}{4}\cdot s\right] = 0$

The coefficient of $\frac{1}{t}$ is:

$$-\frac{s^2}{4}\,2E^2(3+c) + \frac{s^2}{4}(-2E^2(1-c)) + 2E^2\cdot s^2$$

$$= 2E^2\cdot s^2\cdot\left[-\frac{4}{4}+1\right] = 0$$

The coefficient of $\frac{1}{u}$ is:

$$-\frac{s^2}{4}\,2E^2(3-c) + \frac{s^2}{4}(-2E^2(1+c)) + 2E^2 s^2 = 0$$

so

$$i\mathcal{M}(g_R g_R \to g_R g_L) = 0$$

by crossing, also $\quad i\mathcal{M}(g_R g_L \to g_R g_R) = 0$

We have now seen that, of the 16 cases, the 10 cases.

RR → LL	LL → RR
RR → LR, RL	LL → RL, LR
LR, RL → RR	LR, RL → LL

have $\quad i\mathcal{M}(gg \to gg) = 0$

for $\underline{g_R g_R} \to g_R g_R$

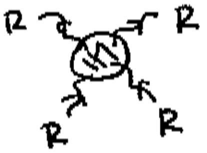
$$\varepsilon_1 \cdot \varepsilon_2 = 1 = \varepsilon_1^* \varepsilon_2^* \qquad \varepsilon_1 \cdot \varepsilon_1^* = -\frac{(1+c)}{2} = \varepsilon_2 \cdot \varepsilon_2^*$$

$$\varepsilon_2 \cdot \varepsilon_1^* = -\frac{(1-c)}{2} = \varepsilon_1 \cdot \varepsilon_2^*$$

$$\varepsilon_1 \varepsilon_1^* + \varepsilon_2 \cdot \varepsilon_2^* + \varepsilon_2 \cdot \varepsilon_1^* + \varepsilon_1 \cdot \varepsilon_2^* - \varepsilon_1^* \varepsilon_2^* - \varepsilon_1 \cdot \varepsilon_2$$

$$= -(1+c) - (1-c) - 1 - 1 = -4$$

then



$$= \frac{-ig^2 f^{abc} f^{cde}}{s} \left\{ t-u + \left(\frac{1+c}{2}\right)^2 s - \left(\frac{1-c}{2}\right)^2 s \right\}$$

$$- \frac{ig^2 f^{ace} f^{bde}}{t} \left\{ \left(\frac{1+c}{2}\right)^2 (s-u) + t - \left(\frac{1-c}{2}\right)^2 t + 8E^2 S^2 \right\}$$

$$- \frac{ig^2 f^{ade} f^{bce}}{u} \left\{ \left(\frac{1-c}{2}\right)^2 (s-t) + u - \left(\frac{1+c}{2}\right)^2 u + 8E^2 S^2 \right\}$$

coeff of $\frac{1}{s}$: $\quad t-u + s\left[\left(\frac{1+c}{2}\right)^2 - \left(\frac{1-c}{2}\right)^2\right]$

$$= -2E^2(1-c) - 2E^2(1+c) + 4E^2 \cdot c$$

$$= 4E^2 c + 4E^2 c$$

$$= 8E^2 c$$

coeff of $\frac{1}{t}$: $\left(\frac{1+c}{2}\right)^2(s-u) + t\left(1 - \left(\frac{1-c}{2}\right)^2\right) + 8E^2s^2$

$$= \frac{2E^2}{4}\left\{(1+c)^2(3+c) - (1-c)\left[4-(1-c)^2\right] + 16s^2\right\}$$

$$= \frac{E^2}{2}\left\{(1+c)^2(3+c) - (1-c)(\underbrace{3+2c-c^2}_{(1+c)(3-c)}) + 16(1-c^2)\right\}$$

$$= \frac{E^2}{2}\left\{(1+c)\right\}\left\{(3+4c+c^2) - (3-4c+c^2) + 16(1-c)\right\}$$

$$= \frac{E^2}{2}(1+c)\left\{8c + 16(1-c)\right\}$$

$$= E^2(1+c)\left\{8-4c\right\}$$

$$= E^2(8+4c-4c^2)$$

$$= E^2(8+4c(1-c))$$

coeff of $\frac{1}{u}$: $\left(\frac{1-c}{2}\right)(s-t) + u\left(1-\left(\frac{1+c}{2}\right)^2\right) + 8E^2s^2$

$=$ above w. $(c \to -c)$

$$= E^2(8-4c(1+c))$$

~ all:

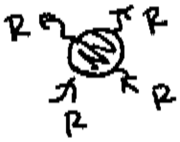
$$= -ig^2 f^{abe}f^{cde}\,(2c)$$
$$-ig^2 f^{ace}f^{bde}\left[-\frac{4}{1-c} - 2c\right]$$
$$-ig^2 f^{ade}f^{bce}\left[-\frac{4}{1+c} + 2c\right]$$

Now, the terms with $(2c)$ form:

$$(2c)\left[f^{abe}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce}\right]$$

$$= (2c)\left[f^{abe}f^{ecd} + f^{bce}f^{ead} + f^{cae}f^{ebd}\right]$$

$$= (2c) \cdot 0 \quad \text{by the Jacobi identity.}$$



$$= +ig^2 f^{ace}f^{bde}\frac{4}{(1-c)} + ig^2 f^{ade}f^{bce}\frac{4}{(1+c)}$$

$$= (-2ig^2)\left(f^{ace}f^{bde}\frac{s}{t} + f^{ade}f^{bce}\frac{s}{u}\right)$$

to compute $g_R g_L \to g_R g_L$, cross $s \leftrightarrow u$, $b \leftrightarrow d$

$$= +2ig^2\left(f^{ace}f^{bde}\frac{u}{t} + f^{abe}f^{cde}\frac{u}{s}\right)$$

let's check this explicitly: for $g_R \underline{g_L \to g_R g_L}$

$$\varepsilon_1\cdot\varepsilon_2 = 0 \qquad \varepsilon_1^*\varepsilon_2^* = 0 \qquad \varepsilon_1\varepsilon_1^* = -\left(\tfrac{1+c}{2}\right) = \varepsilon_2\cdot\varepsilon_2^*$$

$$\varepsilon_1\varepsilon_2^* = \left(\tfrac{1+c}{2}\right) = \varepsilon_2\cdot\varepsilon_1^*$$

$$\varepsilon_1\cdot\varepsilon_1^* + \varepsilon_2\cdot\varepsilon_2^* + \varepsilon_1\cdot\varepsilon_2^* + \varepsilon_2\cdot\varepsilon_1^* - \varepsilon_1\cdot\varepsilon_2 - \varepsilon_1^*\varepsilon_2^* = 0$$

$$= -ig^2 \frac{f^{abe} f^{cde}}{s} \left\{ 0 + \left(\tfrac{1+c}{2}\right)^2 s - \left(\tfrac{1+c}{2}\right)^2 s \right\}$$

$$-ig^2 \frac{f^{ace} f^{bde}}{t} \left\{ \left(\tfrac{1+c}{2}\right)^2 (s-u-t) \right\}$$

$$-ig^2 \frac{f^{ade} f^{bce}}{u} \left\{ \left(\tfrac{1+c}{2}\right)^2 (s-t-u) \right\}$$

$$= -ig^2 f^{ace} f^{bde} \left(\tfrac{1+c}{2}\right)^2 \frac{2s}{t} - ig^2 f^{ade} f^{bce} \left(\tfrac{1+c}{2}\right)^2 \left(\frac{2s}{u}\right)$$

$$= 2ig \left\{ f^{ace} f^{bde} \frac{(1+c)^2}{2} \frac{1}{1-c} + f^{ade} f^{bce} \frac{(1+c)^2}{2} \frac{1}{(1+c)} \right\}$$

the Jacobi identity implies:

$$f^{bce} f^{ade} = f^{ace} f^{bde} - f^{abe} f^{cde}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \tfrac{1}{2} (1+c)^2 \Big(\underbrace{\frac{1}{1-c} + \frac{1}{1+c}}_{\frac{2}{(1-c)(1+c)}} \Big) - f^{abe} f^{cde} \frac{(1+c)^2}{2(1+c)} \right\}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \frac{(1+c)}{(1-c)} + f^{abe} f^{cde} \left[\frac{-\frac{1}{2}(1+c)}{1} \right] \right\}$$

$$= 2ig^2 \left\{ f^{ace} f^{bde} \frac{u}{t} + f^{abe} f^{cde} \frac{u}{s} \right\}$$

as required.

to finish, compute

$$\sum_{\text{colors}} \left| \mathcal{M}(g_R g_R \to s_R \bar{s}_R) \right|^2$$

$$= 4g^4 \Big[\left(\frac{s}{t}\right)^2 f^{ace} f^{bde} f^{ace'} f^{bde'} + \left(\frac{s}{u}\right)^2 f^{ade} f^{bce} f^{ade'} f^{bde'}$$

$$+ 2\frac{s^2}{tu} f^{ace} f^{bde} f^{ade'} f^{bce'} \Big]$$

$$f^{ace} f^{bde} f^{ace'} f^{bde'} = (C_2(G))^2 \delta^{ee'} \delta^{ee'} = d_G (C_2(G))^2 = 8(3)^2$$

$$f^{ace} f^{bde} f^{adf} f^{bcf} = f^{ace} f^{bde} (t^d_G)_{af} (t^b_G)_{cf} \qquad (t^b_G)_{cf} = i f^{cbf}$$

$$= f^{ace} f^{dbe} (t^d_G)_{af} (t^b_G)_{fc}$$

$$= \frac{1}{2} f^{ace} f^{dbe} \left([t^d_G, t^b_G] \right)_{ac}$$

$$= \frac{1}{2} f^{ace} f^{dbe} \, i f^{dbg} (t^g_G)_{ac}$$

$$= \frac{1}{2} C_2(G) f^{acg} \cdot i (i f^{agc})$$

$$= \frac{1}{2} C_2(G) \cdot C_2(G) \cdot d_G = \frac{1}{2} d_G (C_2(G))^2$$

$$= 4 \cdot (72) g^4 \left[\left(\frac{s}{t}\right)^2 + \left(\frac{s}{u}\right)^2 + \frac{s^2}{tu} \right]$$

$$\left(\frac{1}{8}\right)^2 \sum_{\text{colors}} |\mathcal{M}|^2 = \frac{9}{2} g^4 \left[\left(\frac{s}{t}\right)^2 + \left(\frac{s}{u}\right)^2 + \frac{s^2}{tu} \right]$$

$$\frac{d\sigma}{dt}(g_R g_R \to g_R g_R) = \frac{1}{2s}\,\frac{1}{s/2}\,\frac{1}{16\pi}\cdot\frac{9}{2}\,g^4\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\frac{s^2}{tu}\right]$$

$$= \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{s}{u}\right)^2+\left(\frac{s}{t}\right)^2+\frac{s^2}{tu}\right]$$

similarly $$\frac{d\sigma}{dt}(g_L g_L \to g_L g_L) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\frac{s^2}{tu}\right]$$

$$\frac{d\sigma}{dt}(g_R g_L \to g_R g_L) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{u}{t}\right)^2+\left(\frac{u}{s}\right)^2+\frac{u^2}{st}\right]$$

$$= \frac{d\sigma}{dt}(g_L g_R \to g_L g_R)$$

$$\frac{d\sigma}{dt}(g_R g_L \to g_L g_R) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\left(\frac{t}{u}\right)^2+\left(\frac{t}{s}\right)^2+\frac{t^2}{su}\right]$$

$$= \frac{d\sigma}{dt}(g_L g_R \to g_R g_L)$$

for the spin-averaged cross section, add all of these formulae and divide by $\to \frac{1}{2}\cdot\frac{1}{2}$

$$\frac{d\sigma}{dt}(gg\to gg) = \frac{9}{2}\cdot\frac{1}{4}\cdot 2\cdot\frac{\pi\alpha_s^2}{s^2}$$

$$\times\left[\left(\frac{s}{t}\right)^2+\left(\frac{s}{u}\right)^2+\left(\frac{t}{s}\right)^2+\left(\frac{t}{u}\right)^2+\left(\frac{u}{s}\right)^2+\left(\frac{u}{t}\right)^2 + \frac{s^2}{tu}+\frac{t^2}{su}+\frac{u^2}{st}\right]$$

Now, $$\frac{s^2}{tu} = \frac{(-t-u)^2}{tu} = \frac{t}{u} + \frac{u}{t} + 2$$

$$\frac{s^2}{tu} + \frac{t^2}{su} + \frac{u^2}{st} = \frac{t}{u} + \frac{u}{t} + \frac{t}{s} + \frac{s}{t} + \frac{u}{s} + \frac{s}{u} + 6$$

$$= \frac{t+s}{u} + \frac{u+s}{t} + \frac{t+u}{s} + 6$$

$$= -1 - 1 - 1 + 6 = 3$$

$$\frac{s^2}{t^2} + \frac{u^2}{t^2} = \frac{(s+u)^2}{t^2} - \frac{2su}{t^2} = 1 - \frac{2su}{t^2}$$

similarly, $$\frac{s^2}{u^2} + \frac{t^2}{u^2} = 1 - \frac{2st}{u^2} \qquad \frac{t^2}{s^2} + \frac{u^2}{s^2} = 1 - \frac{2tu}{s^2}$$

in all

$$[\ \ ] = 3 + 3 - \frac{2su}{t^2} - \frac{2st}{u^2} - \frac{2tu}{s^2}$$

$$\frac{d\sigma}{dt}(gg \to gg) = \frac{9}{2}\,\frac{\pi\alpha_s^2}{s^2}\left[\,3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2}\right]$$