Physics 331 – Problem Set #3

Solutions

1.) a) Evaluate $\langle q | e^{-\beta H} | q' \rangle$

$$\prod \int_{-\infty}^{\infty} dq_i \; \langle q | e^{-\epsilon H} | q_1 \rangle \langle q_1 | e^{-\epsilon H} | q_2 \rangle \cdots \langle q_{N-1} | e^{-\epsilon H} | q' \rangle$$

Using the method in the text.

$$\langle q_i | e^{-\epsilon H} | q_{i+1} \rangle = \int \frac{dp_i}{2\pi} \, e^{i p_i \cdot (q_i - q_{i+1})} \, e^{-\epsilon H(p, \frac{q_i + q_{i+1}}{2})}$$

assembling the pieces

$$\langle q | e^{-\beta H} | q' \rangle = \int \mathcal{D}q \mathcal{D}p \; e^{\int_0^\beta dt \left[i p \cdot \dot{q} - H(p,q) \right]}$$

for $H = \frac{p^2}{2} + V(q)$ we can eliminate p

by doing the integral

$$\int dp_i \; e^{i p_i (q_i - q_{i+1}) - \epsilon \frac{p_i^2}{2}}$$ complete the square

$$= (\text{const}) \cdot \exp\left[-\frac{1}{2\epsilon} (q_i - q_{i+1})^2 \right]$$

then

$$\langle q | e^{-\beta H} | q' \rangle = \int_{\substack{q(\beta)=q \\ q(0)=q'}} \mathcal{D}q \; e^{-\int_0^\beta dt \, \left(\frac{1}{2}\dot{q}^2 + V(q)\right)}$$

To compute the trace, set $q' = q$ and integrate $\int dq$

$$\mathrm{tr}\, e^{-\beta H} = \int_{q(0)=q(\beta)} \mathcal{D}q \; \exp\left[-\int_0^\beta dt \left[\frac{1}{2}\dot{q}^2 + V(q) \right] \right]$$

b.) Let $q \to x$ $\quad V(q) = \frac{1}{2}\omega^2 x^2$

$$\mathrm{tr}\, e^{-\beta H} = \int \mathcal{D}x(t) \; \exp\left[-\int_0^\beta dt \; \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2\right]\right]$$

$x(t)$ is periodic, so write

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}$$

$x(t)$ is real so $\quad x_{-n} = x_n^*$

the functional integral becomes:

$$\mathrm{tr}\, e^{-\beta H} = C(\beta) \cdot \int dx_0 \prod_{n=1}^{\infty} \int d(\mathrm{Re}\, x_n)\, d(\mathrm{Im}\, x_n)\; e^{-\int_0^\beta dt\, L_E}$$

where

$$\int_0^\beta dt\, L_E = \int_0^\beta dt \;\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2\right)$$

$$= \tfrac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \tfrac{1}{2}\left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega^2\right] |x_n|^2 \cdot 2$$

$$= \tfrac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \left[\left(\tfrac{2\pi n}{\beta}\right)^2 + \omega^2\right]\left[(\mathrm{Re}\, x_n)^2 + (\mathrm{Im}\, x_n)^2\right]$$

so, using

$$\int dx\; e^{-\frac{1}{2}b^2 x^2} = \sqrt{2\pi} \cdot \frac{1}{b}$$

$$\mathrm{tr}[e^{-\beta H}] = C'(\beta) \cdot \frac{1}{\omega} \prod_{n=1}^{\infty} \left(\frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]^{1/2}}\right)^2$$

$$= C''(\beta) \left(\omega \prod_{n=1}^{\infty} \left[1 + \frac{\omega^2\beta^2/4}{(\pi n)^2}\right]\right)^{-1}$$

$$= C'''(\beta) \left(\sinh \omega\beta/2\right)^{-1}$$

$$= C''''(\beta) \left[e^{\omega\beta/2}\left(1 - e^{-\beta\omega}\right)\right]^{-1}$$

$$= \underbrace{C^{(5)}(\beta)}_{\text{various } \omega\text{-independent factors}} e^{-\beta\omega/2} \sum_{m=0}^{\infty} e^{-\beta\omega m}$$

so indeed:

$$\mathrm{tr}\, e^{-\beta H} = \sum_{m=0}^{\infty} e^{-\beta E_m} \qquad E_m = \omega(m+1/2)$$

up to an overall constant independent of ω.

e.) The Hamiltonian of the Klein-Gordon equation is

$$\mathcal{H} = \sum_k h(\omega_k)$$

where h is a Harmonic oscillator Hamiltonian with frequency $\omega_k = (\vec{k}^2+m^2)^{1/2}$. Then it follows from the previous sections that

$$\mathrm{tr}\, e^{-\beta \mathcal{H}} = \int \mathcal{D}\phi \; e^{-\int d^4x\, \mathcal{L}_E}$$

where $\int d^4x\, \mathcal{L}_E = \int d^4x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right]$

and $\phi(t,\vec{x})$ is integrated over configurations periodic in t with periodic β:

$$\phi(t+\beta,\vec{x}) = \phi(t,\vec{x})$$

try this out Fourier decompose:

$$\phi(t,\vec{x}) = \sum_{n\,k} \frac{1}{\sqrt{\beta}\sqrt{V}} e^{i\frac{2\pi n}{\beta}} e^{i\vec{k}\cdot\vec{x}} \phi_{n,k}$$

$$\int d^4x\, \mathcal{L}_E = \sum_{n,k} \tfrac{1}{2}\phi_{-n,-k}\, \phi_{n,k}\, [(\tfrac{2\pi n}{\beta})^2 + \vec{k}^2 + m^2]$$

$$= \sum_{n>0}\sum_{k} |\phi_{n,k}|^2 \, [(\tfrac{2\pi n}{\beta})^2 + \omega_k^2]^{\frac{1}{2}} + \sum_k |\phi_{0,k}|^2 \omega_k^2$$

$$\int D\phi\, e^{-\int d^4x \mathcal{L}_E} = \prod_k \cdot \Big(\omega_k \prod_{n>0} [(\tfrac{2\pi n}{\beta})^2 + \omega^2] \Big)^{-1} \cdot C(\beta)$$

$$= \prod_k \Big(e^{-\beta\omega_k/2} \sum_m e^{-\beta m \omega_k}\Big) \cdot C'(\beta)$$

by the manipulation on the previous page.

For future reference, I will say that this result defines

$$[\det(-\partial^2 + m^2)]^{-\frac{1}{2}}(\beta)$$

d.) Now generalize to fermions. There is an issue with the boundary conditions. Consider

$$\mathrm{tr}[\Psi(\beta)\bar{\Psi}(0)\, e^{-\beta H}] = \mathrm{tr}([e^{\beta H}\Psi(0) e^{-\beta H}]\bar{\Psi}(0)\, e^{-\beta H})$$

$$= \mathrm{tr}(\Psi(0)\, e^{-\beta H}\, \bar{\Psi}(0))$$

this should equal $= \mathrm{tr}\ \bar{\psi}(0)\psi(0)\, e^{-\beta H}$

However, now $\psi, \bar{\psi}$ are in the opposite order:

$$\mathrm{tr}\ \psi(\beta)\bar{\psi}(0)\, e^{-\beta H} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \psi(\beta)\bar{\psi}(0)$$

$$\mathrm{tr}\ \bar{\psi}(0)\psi(0)\, e^{-\beta H} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \bar{\psi}(0)\psi(0)$$

$$= -\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int \mathcal{L}_E}\, \psi(0)\bar{\psi}(0)$$

these are consistent only if

$$\psi(t+\beta) = -\psi(t)$$

Assuming this, evaluate:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-\int dt\, (\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi)}$$

$$\psi(t) = \sum_{p=n+\frac{1}{2}} e^{i\frac{2\pi}{\beta}pt}\psi_p \qquad \bar{\psi}(t) = \sum_{p=n+\frac{1}{2}} e^{i\frac{2\pi}{\beta}pt}\bar{\psi}_p$$

$$\int dt\, L_E = \sum_p \bar{\psi}_{-p}\left(\frac{2\pi i p}{\beta} + \omega\right)\psi_p$$

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-\int L_E} = \prod_p \left(\frac{2\pi i p}{\beta} + \omega\right) \cdot C(\beta)$$

then

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-\int L_E} = \prod_{p>0} \left(\frac{2\pi i p}{\beta} + \omega\right)\left(\frac{-2\pi i p}{\beta} + \omega\right) \cdot C(\beta)$$

$$= \prod_{n=0}^{\infty} \left[\left[\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]^2 + \omega^2\right] \cdot C'(\beta)$$

$$= C''(\beta) \prod_{n=0}^{\infty} \left(1 + \frac{(\beta\omega/2)^2}{(\pi(n+\frac{1}{2}))^2}\right)$$

$$= C''(\beta) \cosh\left(\frac{\beta\omega}{2}\right)$$

$$= C''(\beta) \left(e^{\beta\omega/2} + e^{-\beta\omega/2}\right)$$

then

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-\int L_E} = \sum_{m=0,1} e^{-\beta E_m}$$

with $E_0 = -\omega/2 \qquad E_1 = +\omega/2$

a 2-level system with splitting ω.

e.) Finally, apply this to electrodynamics

$$\int \mathcal{D}A \; e^{-\int d^4x \; +\frac{1}{4}(F_{\mu\nu})^2} \qquad \leftarrow \text{Euclidean action}$$

gauge fix: $\xi = 1$

$$= (\text{const}) \cdot \int \mathcal{D}A \; e^{-\int d^4x \; \frac{1}{2} A_\mu (-\partial^2) A^\mu} \; \underbrace{\det[-\partial^2]}_{\text{Fadeev.-Popov det.}}$$

$$= (\text{const}) \cdot \left(\left(\det[-\partial^2] \right)^{-\frac{1}{2}} \right)^4 \cdot \det(-\partial^2)$$

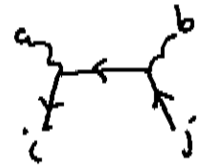
in both cases, over scalar function periodic w. period β.

$$= (\text{const}) \cdot \left(\left[\det(-\partial^2) \right]^{-\frac{1}{2}} \right)^2$$

This is the partition function for $\underline{\underline{2}}$ massless Bose fields.

(The ghosts act as 2 negative degrees of freedom.)

2.) a.) A term contributing to $\underbrace{\bar{q}_i\, q_j}_{\text{color indices}} \to \overbrace{g^a\, g^b}^{G\text{ indices}}$



can be of the form

$$i\mathcal{M} = (i\mathbb{M}) \cdot (t^a t^b)_{ij}$$

then

$$\sum_{ijab} |i\mathcal{M}|^2 = |i\mathbb{M}|^2 \; (t^a t^b)_{ij} \, (t^b t^a)_{ji}$$

$$= |\mathbb{M}|^2 \; \mathrm{tr}\,(t^a t^b t^b t^a)$$

If we average over initial colors, we need:

$$\left(\frac{1}{d_r}\right)^2 \sum_{ij} \sum_{ab} |i\mathcal{M}|^2 = |\mathbb{M}|^2 \cdot \frac{1}{d_r^2}\,\mathrm{tr}(t^a t^b t^b t^a)$$

b.) $$\frac{1}{d_r^2}\,\mathrm{tr}\; t^a t^b t^b t^a = \frac{1}{d_r^2}\,\mathrm{tr}\; t^a t^a t^b t^b = \frac{1}{d_r^2}\,\mathrm{tr}\,(C_2(r))^2 \cdot 1$$

$$= \frac{1}{d_r}\, C_2^2(r)$$

c.) $$\frac{1}{d_r^2}\, \mathrm{tr}[t^a t^b t^a t^b]$$

$$= \frac{1}{d_r^2}\left(\mathrm{tr}[t^a t^b t^b t^a] + \mathrm{tr}\,[t^a t^b\,[t^a, t^b]]\right)$$

$$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2}\,\mathrm{tr}\,[t^a t^b\,(i f^{abc}\, t^c)]$$

$$= \frac{C_2^2(r)}{d_r} + \frac{1}{d_r^2}\,\mathrm{tr}\,[\tfrac{1}{2}[t^a, t^b] \cdot i f^{abc} t^c]$$

$$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2}\,\mathrm{tr}\,[\tfrac{i}{2} f^{abd}\; i f^{abc}\; t^d t^c]$$

now $\quad f^{abd} f^{abc} = C_2(G)\,\delta^{cd}$

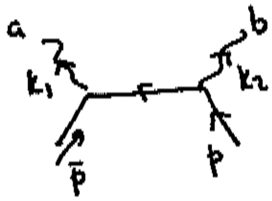
$$= \frac{(C_2(r))^2}{d_r} + \frac{1}{d_r^2}\cdot\left(\frac{-1}{2}\right) C_2(G)\ \mathrm{tr}\ t^c t^c$$

$$= \frac{(C_2(r))^2}{d_r} - \frac{1}{2}\,\frac{1}{d_r}\,C_2(G)\,C_2(r)$$

$$= \frac{1}{d_r}\,C_2(r)\left(C_2(r) - \tfrac{1}{2}C_2(G)\right)$$

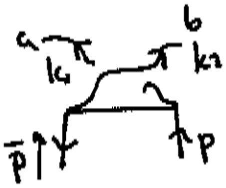
3.) Now compute the cross section for $q_L \bar{q}_R \to gg$
with massless quarks and gluons.

$i\mathcal{M}$ =  +  + 



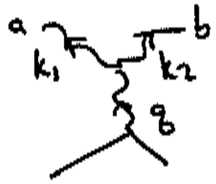
$$= (ig)^2\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_1)\, t^a\, \frac{i\,\gamma\cdot(p-k_2)}{(p-k_2)^2}\, \gamma\cdot\varepsilon^*(k_2)\, t^b\, u(p)$$

$$= +ig^2\, \frac{1}{2p\cdot k_2}\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_1)\, \gamma\cdot(p-k_2)\, \gamma\cdot\varepsilon^*(k_2)\, u(p)\,(t^a t^b)$$



$$= (ig)^2\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_2)\, t^b\, \frac{i\,\gamma\cdot(p-k_1)}{(p-k_1)^2}\, \gamma\cdot\varepsilon^*(k_1)\, t^a\, u(p)$$

$$= +ig^2\, \frac{1}{2p\cdot k_1}\, \bar{v}(\bar{p})\, \gamma\cdot\varepsilon^*(k_2)\, \gamma\cdot(p-k_1)\, \gamma\cdot\varepsilon^*(k_1)\, u(p)\,(t^b t^a)$$



$$= (ig)\, \bar{v}(\bar{p})\, \gamma^\lambda t^c\, u(p)\, \frac{-i}{q^2}\, \varepsilon^*_\mu(k_1)\, \varepsilon^*_\nu(k_2)$$

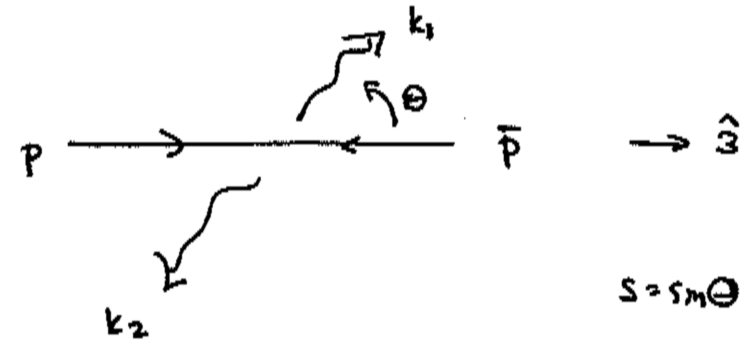
$$g f^{abc}\left[(-k_1+k_2)^\lambda g^{\mu\nu} + g^{\nu\lambda}(-k_2-\underbrace{q}_{k_1+k_2})^\mu + g^{\lambda\mu}(q+k_1)^\nu\right]$$

$$= g^2 f^{abc}\, \bar{v}(\bar{p})\, \gamma^\lambda\, t^c\, u(p)\, \frac{1}{q^2}$$

$$\left[(k_2-k_1)^\lambda\, \varepsilon^*_\mu(k_1)\varepsilon^\mu(k_2) - 2\, k_2\cdot\varepsilon^*(k_1)\, \varepsilon^{*\lambda}(k_2) + 2\, k_1\cdot\varepsilon^*(k_2)\, \varepsilon^{\lambda *}(k_1)\right]$$

using $\quad k_1\cdot\varepsilon^*(k_1) = k_2\cdot\varepsilon^*(k_2) = 0$

set up the kinematics as



$s = \sin\theta \qquad c = \cos\theta$

$$p = (E, 0, 0, E) \qquad k_1 = (E, Es, 0, Ec)$$
$$\bar p = (E, 0, 0, -E) \qquad k_2 = (E, -Es, 0 -Ec)$$

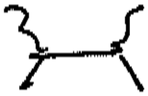
$$2p\cdot k_2 = 2E^2(1+c) \qquad q^2 = 4E^2$$
$$2p\cdot k_1 = 2E^2(1-c)$$

$$(p-k_2) = (0, Es, 0, E(1+c))$$
$$(p-k_1) = (0, -Es, 0, E(1-c))$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \vec\sigma) \qquad \bar\sigma^\mu = (1, -\vec\sigma)$$

Evaluate the diagrams for $\quad q_L \bar q_R \to gg$

$$u_L(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix} \qquad \bar v_R(\bar p) = \sqrt{2E}\,(0\;0 \mid -1\;0)$$



$$= ig^2 \frac{1}{2E^2(1+c)} \sqrt{2E}\,(0\,0\,|\,{-1}\,0) \begin{pmatrix} & \sigma\cdot\varepsilon_1^* \\ \bar\sigma\cdot\varepsilon_1^* & \end{pmatrix} \cdot \begin{pmatrix} & \sigma\cdot(p-k_2) \\ \bar\sigma\cdot(p-k_2) & \end{pmatrix} \begin{pmatrix} & \sigma\cdot\varepsilon_2^* \\ \bar\sigma\cdot\varepsilon_2^* & \end{pmatrix} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; t^a t^b$$

$$= \frac{ig^2}{E(1+c)} \,(-1\,0)\; \bar\sigma\cdot\varepsilon_1^* \;\sigma\cdot(p-k_2)\; \bar\sigma\cdot\varepsilon_2^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; t^a t^b$$

$$= \frac{ig^2}{E(1-c)} \,(-1\,0)\; \bar\sigma\cdot\varepsilon_2^* \;\sigma\cdot(p-k_1)\; \bar\sigma\cdot\varepsilon_1^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; t^b t^a$$

$$= g^2 f^{abc} \frac{2E}{4E^2} \,(-1\,0)\; \bar\sigma^\lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; t^c \cdot \left[(k_2-k_1)^\lambda \,\varepsilon_1^*\cdot\varepsilon_2^* - 2\, k_2\,\varepsilon^*(k_1)\, \varepsilon_2^{*\lambda} + 2 k_1\cdot\varepsilon_2^* \,\varepsilon_1^{*\lambda} \right]$$

$$= \frac{g^2 f^{abc}}{2E} \, t^c \,(-1\,0) \left[\bar\sigma\cdot(k_2-k_1)\, \varepsilon_1^*\cdot\varepsilon_2^* - 2\, k_2\cdot\varepsilon_1^*\, \varepsilon_2^*\cdot\bar\sigma + 2\, k_1\cdot\varepsilon_2^*\, \varepsilon_1^*\cdot\bar\sigma \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to go further we need the explicit forms for

$$\sigma\cdot(p-k_2) = \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix}$$

$$\sigma\cdot(p-k_1) = \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix}$$

$$\bar{\sigma}\cdot(k_2-k_1) = \begin{pmatrix} -2Ec & -2Es \\ -2Es & 2Ec \end{pmatrix}$$

and for the polarization vectors:

$$\varepsilon^*_{1R} = \frac{1}{\sqrt{2}}(0, c, -i, -s) \qquad \bar{\sigma}\cdot\varepsilon^*_{1R} = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix}$$

$$\varepsilon^*_{1L} = \frac{1}{\sqrt{2}}(0\ c, +i, -s) \qquad \bar{\sigma}\cdot\varepsilon^*_{1L} = \frac{1}{\sqrt{2}}\begin{pmatrix} -s & c+1 \\ c-1 & s \end{pmatrix}$$

$$\varepsilon^*_{2R} = \frac{1}{\sqrt{2}}(0, -c, -i, s) \qquad \bar{\sigma}\cdot\varepsilon^*_{2R} = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -c-1 \\ -c+1 & -s \end{pmatrix}$$

$$\varepsilon^*_{2L} = \frac{1}{\sqrt{2}}(0, -c, +i, s) \qquad \bar{\sigma}\cdot\varepsilon^*_{2L} = \frac{1}{\sqrt{2}}\begin{pmatrix} s & -c+1 \\ -c-1 & -s \end{pmatrix}$$

Note that $\quad k_1\cdot\varepsilon^*_1 = 0 = k_2\cdot\varepsilon^*_2$

and also that $\quad k_2\cdot\varepsilon^*_1 = 0 = k_1\cdot\varepsilon^*_2$

$$\varepsilon^*_{1R}\cdot\varepsilon^*_{2R} = 1 = \varepsilon^*_{1L}\cdot\varepsilon^*_{2L}$$

$$\varepsilon^*_{1R}\cdot\varepsilon^*_{2L} = 0 = \varepsilon^*_{1L}\cdot\varepsilon^*_{2R}$$

Now evaluate the diagrams on p. 13 for the case

$$q_L \bar{q}_R \longrightarrow g_R g_R$$

$$= \frac{ig^2}{E(1+c)} \,(-1\ \ 0)\, \frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} s & -(c+1) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad t^a t^b$$

$$= \frac{ig^2}{(1+c)} \frac{1}{2E} (s \quad 1-c) \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix} \begin{pmatrix} -(1+c) \\ -s \end{pmatrix} \quad t^a t^b$$

$[(1-c)(1+c) = s^2]$

$$= \frac{ig^2}{1+c} \cdot \frac{E}{2E} \left\{ s(1+c)^2 + s^3 + s^3 - s^3 \right\} t^a t^b$$

$$= \frac{ig^2}{1+c} \frac{1}{2} s \quad (1+c) \left[(1+c) + (1-c) \right] \quad t^a t^b$$

$$= ig^2 \; s \frac{(1+c)}{(1+c)} \; t^a t^b \quad = \quad ig^2 s \quad t^a t^b$$

$$= \frac{ig^2}{E(1-c)} \quad (-1\ \ 0) \quad \frac{1}{\sqrt{2}} \begin{pmatrix} s & -(c+1) \\ 1-c & -s \end{pmatrix} \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad t^b t^a$$

$$= \frac{ig^2}{2(1-c)E} \begin{pmatrix} -s & c+1 \end{pmatrix} \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix} \begin{pmatrix} c-1 \\ s \end{pmatrix} t^b t^a$$

$$= \frac{ig^2}{2(1-c)} \cdot \frac{E}{2E} \cdot \left\{ -s(1-c)^2 - s^3 + \underbrace{s(c+1)(c-1)}_{-s^3} + s(s^2) \right\} \cdot t^b t^a$$

$$= \frac{ig^2}{(1-c)} \frac{1}{2} \left\{ -s(1-c)^2 - s(1-c)(1+c) \right\} t^b t^a$$

$$= + \frac{ig^2}{(1-c)} \left[-s(1-c) \right] \left[\frac{(1-c) + (1+c)}{2} \right] t^b t^a$$

$$= \underline{-ig^2} \, s \; t^b t^a$$

$$= \frac{g^2 f^{abc}}{2E} t^c \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} -2Ec & -2Es \\ -2Es & 2Ec \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \varepsilon^*_{1R} \cdot \varepsilon^*_{2R}$$

$$= \frac{g^2 f^{abc}}{2E} t^c \; 2Es \cdot 1$$

$$= g^2 f^{abc} t^c \, s$$

adding the three pieces

$$i\mathcal{M}(q_L \bar{q}_R \to g_R g_R) = +ig^2 s\, t^a t^b - ig^2 s\, t^b t^a + g^2 f^{abc} t^c s$$

$$= ig^2 s \underbrace{[t^a, t^b]}_{i f^{abc} t^c} + g^2 s\, f^{abc} t^c$$

$$= 0$$

the amplitude for $q_L \bar{q}_R \to g_L g_L$ has a similar cancellation.
(by CP)

$$i\mathcal{M}(q_L \bar{q}_R \to g_L g_L) = 0$$

Finally, consider $q_L \bar{q}_R \to g_R(k_1)\, g_L(k_2)$

$$= \frac{ig^2}{E(1+c)} \begin{pmatrix} -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix}$$

$$\cdot \frac{1}{\sqrt{2}} \begin{pmatrix} s & -c+1 \\ -(c+1) & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^a t^b$$

$$= \frac{ig^2}{E(1+c)} \frac{1}{2} \begin{pmatrix} s & (1-c) \end{pmatrix} \begin{pmatrix} -E(1+c) & -Es \\ -Es & E(1+c) \end{pmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix} t^a t^b$$

$$= \frac{ig^2}{1+c} \; \frac{E}{2E} \left\{ -s(1+c)(1-c) + s^3 - s(1-c)^2 - s(s^2) \right\} t^a t^b$$

$$= \frac{ig^2}{1+c} \cdot \frac{1}{2} \left\{ -s(1-c)(1+c) - s(1-c)^2 \right\} t^a t^b$$

$$= \frac{ig^2}{1+c} \cdot \frac{1}{2} \; -s(1-c) \left[(1+c) + (1-c) \right] t^a t^b$$

$$= \frac{-ig^2}{1+c} \; s(1-c) \; t^a t^b$$

$$= \frac{ig^2}{E(1-c)} \begin{pmatrix} -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} s & -c+1 \\ -(c+1) & -s \end{pmatrix} \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix}$$

$$\cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -s & c-1 \\ c+1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; t^b t^a$$

$$= \frac{ig^2}{E(1-c)} \; \frac{1}{2} \; \begin{pmatrix} -s & -(1-c) \end{pmatrix} \begin{pmatrix} -E(1-c) & Es \\ Es & E(1-c) \end{pmatrix} \begin{pmatrix} -(1-c) \\ s \end{pmatrix} \; t^b t^a$$

$$= \frac{ig^2}{1-c} \; \frac{E}{2E} \left\{ -s(1-c)^2 - s^3 + s(1-c)^2 - s(1-c)^2 \right\} t^b t^a$$

$$= \frac{ig^2}{1-c} \; \frac{1}{2} \left\{ -s(1-c) \left[(1-c) + (1+c) \right] \right\} t^b t^a$$

$$= \frac{ig^2}{1-c}\,[-s(1-c)]\,t^b t^a = -ig^2 s\; t^b t^a$$

and

 $= 0$ since $\varepsilon^*_{1R}\cdot\varepsilon^*_{2L} = 0$

then

$$i\mathcal{M}(q_L\bar{q}_R \to g_{1R}\,g_{2L}) = (-ig^2 s)\left(\frac{1-c}{1+c}\,t^a t^b + t^b t^a\right)$$

now square and sum over colors

$$\frac{1}{d_r^2}\sum_{\text{color}} |i\mathcal{M}|^2 = g^4 s^2 \Big\{ \frac{(1-c)^2}{(1+c)^2}\cdot\frac{1}{d_r}(C_2(r))^2$$

$$+ \quad 1 \quad \cdot\ \frac{1}{d_r}(C_2(r))^2$$

$$+ \quad 2\,\frac{(1-c)}{(1+c)}\cdot\frac{1}{d_r}\,C_2(r)\left(C_2(r) - \frac{C_2(G)}{2}\right)\Big\}$$

$$= g^4 s^2 \cdot\left[\frac{1}{d_r}\,C_2(r)\right]\cdot\Big\{\left[\frac{1-c}{1+c} + 1\right]^2 C_2(r)$$

$$- \frac{(1-c)}{1+c}\,C_2(G)\Big\}$$

$$= \quad g^4 s^2 \left(\frac{1}{d_r} C_2(r)\right) \left\{ \frac{4}{(1+c)^2} \; C_2(r) \right.$$

$s^2 = (1+c)(1-c)$

$$\left. - \frac{(1-c)}{(1+c)} \; C_2(G) \right\}$$

$$= \quad g^4 \; \frac{1}{d_r} C_2(r) \cdot \left\{ 4\, C_2(r) \cdot \left(\frac{1-c}{1+c}\right) \right.$$

$$\left. - (1-c)^2 C_2(G) \right\}$$

put in

$$s = 4E^2 \qquad t = -2E^2(1-c) \qquad u = -2E(1+c)$$

$$= \quad g^4 \; \frac{1}{d_r} C_2(r) \left\{ 4 C_2(r) \frac{t}{u} - 4 C_2(G) \frac{t^2}{s^2} \right\}$$

$$\frac{d\sigma}{d\cos\theta} \left(q_L \bar{q}_R \to g_R(k_1) g_L(k_2)\right)$$

$$= \frac{1}{2s} \frac{1}{16\pi} \; g^4 \; \frac{C_2(r)}{d_r} \cdot 4 \left(C_2(r) \frac{t}{u} - C_2(G) \frac{t^2}{s^2} \right)$$

$$= \quad \frac{2\pi \, (g^2/4\pi)^2}{s} \; \frac{C_2(r)}{d_r} \left(C_2(r) \frac{t}{u} - C_2(G) \frac{t^2}{s^2} \right)$$

to include $g_L(k_1)\, g_R(k_2)$

add the formula with $t \leftrightarrow u$

to average over initial spins multiply by $2 \times \frac{1}{4}$

$$\frac{d\sigma}{d\cos\theta}(q\bar{q} \to gg) = \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{d_r}$$

$$\cdot \left[C_2(r)\left(\frac{t}{u} + \frac{u}{t}\right) - C_2(G)\, \frac{t^2+u^2}{s^2} \right]$$

$$= \frac{\pi}{s}\left(\frac{g^2}{4\pi}\right)^2 \frac{C_2(r)}{d_r}$$

$$\cdot \left[C_2(r)\left(\frac{2}{1-\cos^2\theta}\right) - \frac{C_2(G)}{2}(1+\cos^2\theta) \right]$$

this formula is to be integrated over

$\cos\theta > 0$ only

(identical particles)