Physics 331 – Problem Set #2

Solutions

1.) a.) $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi$

where $D_\mu = \partial_\mu + ieA_\mu$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$
$$+ \partial_\mu\phi^*(ieA_\mu\phi) - ieA_\mu\phi^*\partial^\mu\phi + e^2A_\mu A^\mu\phi^*\phi$$

We need to show, first, that the propagator of the ϕ field is

$$\phi\phi^* = \int\frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\frac{i}{k^2-m^2+i\epsilon}$$

Using $\mathcal{L}_0 = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi = \phi^*(-\partial^2-m^2)\phi$

$$Z[J,J^*] = \int\mathcal{D}\phi\,\mathcal{D}\phi^*\,\exp\left[i\int d^4x\,(\mathcal{L}_0 + J^*\phi + \phi^*J)\right]$$

$$\langle\phi(x)\phi^*(y)\rangle = \left(-i\frac{\delta}{\delta J^*(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)Z \,/\, Z\,\Big|_{J=J^*=0}$$

now

$$Z[J,J^*] = \int\mathcal{D}\phi\,\mathcal{D}\phi^*\,\exp\Big[i\int d^4x\,\left[\phi^*(x) + \int d^4z\,J^*(z)\,G(z,x)\right](-\partial_x^2-m^2)$$
$$\cdot\left[\phi(x) + \int d^4y\,G(x,y)J(y)\right]$$
$$-i\int d^4x\,d^4y\,\phi^*(x)\,G(x,y)\,\phi(y)\Big]$$

where $(-\partial_x^2-m^2)G(x,y) = \delta(x-y)$

then, with $\phi'(x) = \phi(x) + \int d^4y\, G(x,y)\, J(y)$

$$\phi^{*\prime}(x) = \phi'^{*}(x) + \int d^4z\, J^*(z)\, G(z,x)$$

$$\mathcal{D}\phi'\mathcal{D}\phi^{*\prime} = \mathcal{D}\phi\,\mathcal{D}\phi^*$$

$$Z[J,J^*] = Z[0,0]\ \exp\left[-i \int d^4x\, d^4y\ \phi^*(x)\, G(x,y)\, \phi(y)\right]$$

$$\langle \phi(x)\phi^*(y)\rangle = (-i)^2(-i)\ G(x,y) = i\,G(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{i}{k^2-m^2}$$

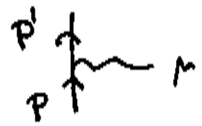
as promised.

The interaction vertices follow from

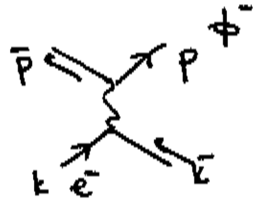
$$i\Delta\mathcal{L} = i\left[\partial_\mu\phi^*\, ieA^\mu\phi - ieA_\mu\phi^*\partial^\mu\phi + e^2A_\mu A^\mu\phi^*\phi\right]$$

$$= eA_\mu\,(\underbrace{-\partial^\mu\phi^*}_{-ip'^\mu})\,\phi + eA_\mu\,\phi^*\underbrace{\partial^\mu}_{-ip^\mu}\phi + ie^2A_\mu A^\mu\phi^*\phi$$

for 

so  $= -ie\,(p+p')^\mu$  $= 2ie\, g^{\mu\nu}$

b.)



$$i\mathcal{M} = \left[-ie\,\bar v(\bar k)\gamma_\mu u(k)\right]\frac{-i}{q^2}\left[-ie(p-\bar p)_\mu\right]$$

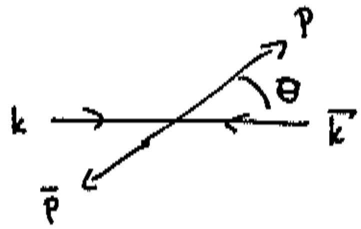
$$= \frac{ie^2}{q^2}\,\bar v(\bar k)\gamma_\mu u(k)\,(p-\bar p)_\mu$$

$$\frac{1}{4}\sum_{\text{pols}}|\mathcal{M}|^2 = \frac{1}{4}\,\frac{e^4}{(q^2)^2}\,\mathrm{tr}\left[\bar{\not k}\,\gamma_\mu\,\not k\,\gamma_\nu\right](p-\bar p)^\mu(p-\bar p)^\nu$$

$$= \frac{1}{4}\,\frac{e^4}{(q^2)^2}\cdot 4\left[\bar k^\mu k^\nu + \bar k^\nu k^\mu - g^{\mu\nu}\,\bar k\cdot k\right](p-\bar p)_\mu(p-\bar p)_\nu$$

$$= \frac{e^4}{(q^2)^2}\left[2\left[\bar k\cdot(p-\bar p)\;k\cdot(p-\bar p)\right] - (p-\bar p)^2\,\bar k\cdot k\right]$$

kinematics :



$k = (E, 0, 0, E)$ $\qquad p = (E, p\sin\theta, 0, p\cos\theta)$

$\bar k = (E, 0, 0, -E)$ $\qquad \bar p = (E, -p\sin\theta, 0, -p\cos\theta)$

$$p - \bar p = 2p\,(0, \sin\theta, 0, \cos\theta)$$

$k\cdot(p-\bar p) = -2pE\cos\theta$ $\qquad k\cdot\bar k = 2E^2$ $\qquad q^2 = 4E^2$

$\bar k\,(p-\bar p) = +2pE\cos\theta$ $\qquad (p-\bar p)^2 = -4p^2$

so (above) $= \dfrac{e^4}{(q^2)^2}\cdot 2\cdot 4E^2p^2\,(-\cos^2\theta + 1)$

$$\frac{1}{4}\sum_{\text{pols}} |\mathcal{M}|^2 = \frac{e^4}{2}\ \frac{p^2}{E^2}\ \sin^2\theta$$

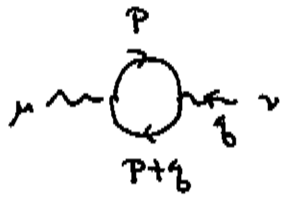
$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to \phi^+\phi^-) = \frac{1}{2s}\left[\frac{1}{16\pi}\frac{p}{E}\right]\frac{e^4}{2}\ \frac{p^2}{E^2}\ \sin^2\theta$$

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to \phi^+\phi^-) = \frac{\pi\alpha^2}{4s}\left(\frac{p}{E}\right)^3 \sin^2\theta$$

$$\int_{-1}^{1} d\cos\theta\ \sin^2\theta = \frac{4}{3}$$

$$\sigma(e^+e^- \to \phi^+\phi^-) = \frac{\pi\alpha^2}{3s}\left(\frac{p}{E}\right)^3$$

for $E \gg m$	$d\sigma/d\cos\theta$		$\sigma_{tot.}$
$e^+e^- \to \mu^+\mu^-$	$\sim(1+\cos^2\theta)$	$\to$	$\frac{4\pi\alpha^2}{3s}$
$e^+e^- \to \phi^+\phi^-$	$\sim(1-\cos^2\theta)$	$\to$	$\frac{\pi\alpha^2}{3s}$

c.) 

+ 

$$= (-ie)^2 \int \frac{d^4p}{(2\pi)^4}\ (2p+q)^\mu \frac{i}{p^2-m^2}\ (2p+q)^\nu \frac{i}{(p+q)^2-m^2}$$

$$= e^2 \int_0^1 dx \int \frac{d^4\mathbb{P}}{(2\pi)^4}\ \frac{1}{[\mathbb{P}^2+x(1-x)q^2-m^2]^2}\ \left(2\mathbb{P}+(1-2x)q\right)^\mu\left(2\mathbb{P}+(1-2x)q\right)^\nu$$

$$\mathbb{P} = p + xq$$

$$p = \mathbb{P} - xq \qquad (2p+q) = 2\mathbb{P} + (1-2x)q$$

$$p+q = \mathbb{P} + (1-x)q$$

$$= e^2 \int_0^1 dx \cdot \frac{i}{(4\pi)^{d/2}} \left\{ - \frac{\Gamma(1-d/2)}{[m^2 - x(1-x)q^2]^{1-d/2}} \cdot 4 \cdot \frac{g^{\mu\nu}}{2} \right.$$

$$\left. + \frac{\Gamma(2-d/2)}{[m^2 - x(1-x)q^2]^{2-d/2}} (1-2x)^2 q^\mu q^\nu \right\}$$

$$= 2ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} \; g^{\mu\nu}$$

$$= 2ie^2 i g^{\mu\nu} \int \frac{d^4p}{(2\pi)^4} \frac{(p+q)^2-m^2}{(p^2-m^2)((p+q)^2-m^2)}$$

$$= -2e^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^4 P}{(2\pi)^4} \frac{1}{[P^2+x(1-x)q^2-m^2]^2} [(P+(1-x)q)^2 - m^2]$$

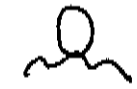
$$= -2e^2 g^{\mu\nu} \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \left\{ - \frac{\Gamma(1-d/2)}{(m^2-x(1-x)q^2)^{1-d/2}} \frac{d}{2} \right.$$

$$\left. + \frac{\Gamma(2-d/2)}{[m^2-x(1-x)q^2]^{2-d/2}} [(1-x)^2 q^2 - m^2] \right\}$$

actually, we can symmetrize!

$$(1-x)^2 \to \frac{1}{2}[(1-x)^2 + x^2] = \frac{1}{2}[1-2x+2x^2]$$

the sum of the diagrams is!

 + 

$$= \frac{ie^2}{(4\pi)^{d/2}} \int_0^1 dx \left\{ - \frac{\Gamma(1-d/2)}{[m^2-x(1-x)q^2]^{1-d/2}} g^{\mu\nu} (2-d) \right.$$

$$\left. + \frac{\Gamma(2-d/2)}{[m^2-x(1-x)q^2]^{2-d/2}} \left((1-2x)^2 q^\mu q^\nu - (1-2x+2x^2) q^2 g^{\mu\nu} + 2m^2 g^{\mu\nu} \right) \right\}$$

the first term is :

$$\frac{\Gamma(1-d/2)}{[m^2-x(1-x)q^2]^{1-d/2}}\,2-d \;=\; \frac{2\cdot\Gamma(2-d/2)}{[m^2-x(1-x)q^2]^{2-d/2}}\cdot[m^2-x(1-x)q^2]$$

so

 + 

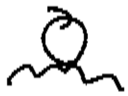
$$= \frac{ie^2}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 dx\,\frac{1}{[m^2-x(1-x)q^2]^{2-d/2}}$$

$$\{\; -2g^{\mu\nu}(m^2-x(1-x)q^2) + 2m^2 g^{\mu\nu} - (1-2x+2x^2)\,q^2 g^{\mu\nu} + (1-2x)^2 q^\mu q^\nu \;\}$$

$$\{\} = (-1)\left[(1-4x+4x^2)\,q^2 g^{\mu\nu} - (1-2x)^2 q^\mu q^\nu\right]$$

$$= (-1)\,(1-2x)^2\; q^2 g^{\mu\nu} - q^\mu q^\nu$$

so

 + 

$$= \frac{-ie^2}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 dx\,\frac{(1-2x)^2}{[m^2-x(1-x)q^2]^{2-d/2}}\cdot(q^2 g^{\mu\nu}-q^\mu q^\nu)$$

$d\to 4$

$$= \frac{-i\,e^2}{(4\pi)^2}\,(q^2 g^{\mu\nu}-q^\mu q^\nu)\int_0^1 dx\,(1-2x)^2\,\lg\left(\frac{\Lambda^2}{m^2-x(1-x)q^2}\right)$$

$$\int_0^1 dx\,(1-2x)^2 = 1-4\cdot\tfrac{1}{2}+4\,\tfrac{1}{3} = \tfrac{1}{3}$$

for an electron positron or $\mu^+\mu^-$ loop, we found ~ 330:

$$= -\frac{8ie^2}{(4\pi)^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 dx\; x(1-x)\, \log \frac{\Lambda^2}{m^2 - x(1-x)q^2}$$

$$\int dx\; x(1-x) = \frac{1}{6}$$

for $q^2 \gg m^2$

$$= -i\,\frac{e^2}{(4\pi)^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot C \cdot \log \frac{\Lambda^2}{q^2}$$

$$C = \begin{cases} \frac{4}{3} & \mu^+\mu^- \\ \frac{1}{3} & \phi^*\phi \end{cases}$$

2.) • Let Φ be a linear combination of free fields:

$$\Phi = \int dx\; f(x)\, \phi(x)$$

a.) $$\langle \Phi^2 \rangle = \int dx\, dy\; f(x)\, f(y)\; \overbrace{\phi(x)\, \phi(y)} \equiv \overbrace{\Phi\,\Phi}$$

then

$$\langle \Phi^4 \rangle = \langle \Phi\Phi\Phi\Phi \rangle$$

$$= \overbrace{\Phi\Phi}\,\overbrace{\Phi\Phi} + \Phi\Phi\Phi\Phi + \Phi\Phi\Phi\Phi$$

$$= 3\left(\langle \overbrace{\Phi\Phi} \rangle\right)^2$$

similarly

$$\langle \Phi^{2n} \rangle = (\text{\# of contractions}) \cdot (\overline{\Phi\Phi})^n$$

$$= (2n-1)(2n-3) \cdots 1$$

then

$$\langle e^{\Phi} \rangle = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n \right\rangle$$

terms w. n = odd vanish

$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!} (\langle \Phi^2 \rangle)^m \; (2m-1)(2m-3) \cdots 1$$

$$= \sum_{m=0}^{\infty} \frac{1}{2m\,(2m-2)(2m-4)\cdots} (\langle \Phi^2 \rangle)^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{1}{m!} (\langle \Phi^2 \rangle)^m$$

$$= e^{\langle \Phi^2 \rangle / 2}$$

b) With functional integration:

$$\langle e^{\Phi} \rangle = \frac{\int \mathcal{D}\phi \; e^{i \int d^4x \, \mathcal{L}_0(\phi)} \; e^{\int d^4x \, f(x)\, \phi(x)}}{\int \mathcal{D}\phi \; e^{i \int \mathcal{L}_0}}$$

if $\mathcal{L}_0 = \frac{1}{2} \phi \Delta \phi$ where e.g. $\Delta = (-\partial^2 - m^2)$

$$\overline{\phi\phi} = i \Delta^{-1}(x,y)$$

the exponent of the numerator above is

$$i\int d^4x \;\left[\tfrac{1}{2}\phi\,\Delta\,\phi + (-if)\cdot\phi\right]$$

$$= i\int d^4x \left(\tfrac{1}{2}\,(\phi - i f\Delta^{-1})\,\Delta\,(\phi - i\Delta^{-1}f) + \tfrac{1}{2} f\,\Delta^{-1} f\right)$$

then

$$\langle e^{\Phi}\rangle = \exp\left[\tfrac{1}{2}\int dx\,dy\; f(x)\; i\Delta^{-1}(x,y)\; f(y)\right]$$

$$= \exp\left[\tfrac{1}{2}\int dx\,dy\; f(x)\; \overline{\phi(x)\,\phi(y)}\; f(y)\right]$$

$$= \exp\left[\tfrac{1}{2}\,\langle \Phi^2\rangle\right]$$

3.) a.) Using this result

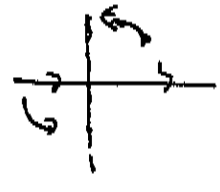
$$\left\langle e^{-ie\oint_P dx_\mu A^\mu}\right\rangle = \exp\left[-\frac{e^2}{2}\oint_P dx^\mu \oint_P dy^\nu \,\langle A_\mu(x)\,A_\nu(y)\rangle\right]$$

Now

$$\langle A_\mu(x)\,A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i g^{\mu\nu}}{k^2 + i\epsilon}$$

to evaluate the integral, rotate to Euclidean space.

$$x^0, y^0 = -i x_E^0,\; -i y_E^0 \qquad k^0 = i k_E^0$$



then

$$\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\varepsilon} e^{-ik\cdot(x-y)} \longrightarrow \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{-k_E^2} e^{-ik_E(x_E-y_E)}$$

this is the solution of

$$\nabla^2 \Delta_E(x_E, y_E) = \delta^{(4)}(x_E-y_E) \quad \text{in 4-d Euclidean space}$$

let $\quad \vec{E} = \vec{\nabla} \Delta_E(r) \quad$ then

$$\int d^4x \; \vec{\nabla}\cdot\vec{E} = \int d^3S \; \hat{r}\cdot\vec{E} = 1$$

so $$\vec{E} = \frac{\hat{r}}{(\text{Area of sphere})} = \frac{\hat{r}}{2\pi^2 r^3}$$

$$\Delta_E = -\frac{1}{4\pi^2 r^2}$$

rotate back to Minkowski space

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} = \frac{1}{4\pi^2 (x-y)^2}$$

then

$$\left\langle e^{-ie\oint_P dx_\mu A^\mu} \right\rangle = \exp\left[-e^2 \oint_P dx_\mu \oint_P dy_\mu \frac{1}{8\pi^2(x-y)^2}\right]$$

$$\parallel$$

$$\langle U_P \rangle$$

b.) For a loop of the form:

$$\langle U_P \rangle = (\text{R-independent factor})$$

$$\cdot \exp\Big[-e^2 \cdot 2 \cdot \int_{-T/2}^{T/2} dx^0 \underbrace{\int_{T/2}^{-T/2}}_{\substack{\text{opposite} \\ \text{direction}}} dy^0 \, \frac{1}{8\pi^2 \left[(x^0-y^0)^2 - R^2 - i\varepsilon \right]} \Big]$$

$$\cong (\text{R-indep}) \cdot \exp\Big[+ \frac{e^2}{4\pi^2} T \int_{-\infty}^{\infty} dx^0 \, \frac{1}{\left[(x^0)^2 - R^2 - i\varepsilon \right]} \Big]$$

to evaluate the integral, rotate the contour:

$$\int dx^0 \frac{1}{(x^0)^2 - R^2} = -i \int dx_E^0 \, \frac{1}{-\left((x_E^0)^2 + R^2 \right)} = i \cdot \frac{\pi}{R}$$

so

$$\langle U_P \rangle = (\text{const}) \cdot \exp\Big[+ i \, \frac{e^2}{4\pi R} \, T \Big]$$

comparing to $\langle U_P \rangle = (\text{const}) \cdot e^{-i V(R) T}$

$$V(R) = - \frac{e^2}{4\pi R}$$

e.) In a non-Abelian gf theory

$$\langle U_P \rangle = \langle \mathrm{tr} \left(\mathcal{P} \exp\left[+ig \oint dx_\lambda A^a_\lambda t^a_r \right] \right) \rangle$$

$$= \mathrm{tr} \left(1 - g^2 \oint dx_\lambda \, dy_\nu \, A^a_\lambda A^b_\nu \, t^a_r t^b_r + \cdots \right)$$

$\mathrm{tr}\, 1 = d_r$

$A^a_\lambda A^b_\nu$ is $\propto \delta^{ab}$, so the second term includes $\mathrm{tr}\, t^a_r t^a_r = C_2(r) d_r$

so $$= d_r \left(1 + \frac{i g^2 C_2(r)}{4\pi R} T + \cdots \right)$$

If this exponentiates (it does)

$$\langle U_P \rangle = (\mathrm{const.}) \cdot \exp\left[+i \frac{g^2 C_2(r)}{4\pi R} T \right]$$

or $$V(R) = -\frac{g^2 C_2(r)}{4\pi R}$$

Both answers for V(R) agree with Problem 2 of the previous problem set.

3.) a.) Let $D(x,y,T)$ satisfy:

$$\left[i\frac{\partial}{\partial T} - (\partial^2+m^2) \right] D(x,y,T) = i\,\delta(T)\,\delta^{(4)}(x-y)$$

$D(x,y,T)$ is the propagator of the Schrödinger equation

$$\langle x|\, e^{-iHT}\, |y\rangle \quad \text{with} \quad H = [-\nabla^2 + \partial_0^2 + m^2]$$

for $\hat{H} = (-\nabla^2 + m^2)$ we would have

$$\langle x|\, e^{-i\hat{H}T}\, |y\rangle = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, e^{-i(|\vec{k}|^2+m^2)T}$$

similarly, in 4 dimensions

$$\langle x|\, e^{-iHT}\, |y\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, e^{-i(|\vec{k}|^2-(k^0)^2+m^2)T}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, e^{i(k^2-m^2)T}$$

$$\int_0^\infty dT\; D(x,y,T) = \int_0^\infty dT\; \langle x|\, e^{-iHT}\, |y\rangle$$

$$= \int_0^\infty dT \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, e^{i(k^2-m^2)T}\, e^{-\epsilon T}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{1}{-i(k^2-m^2)+\epsilon}$$

so that, finally,

$$\int_0^\infty dT \; D(x,y,T) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik\cdot(x-y)} \; \frac{i}{k^2 - m^2 + i\varepsilon}$$

Now, the solution of the Schrödinger eq has a functional integral representation:

$$\langle x | \, e^{-i\hat{H}T} \, | y \rangle = \int_{\substack{X(T) = x^\mu \\ X(0) = y^\mu}} \mathcal{D}X(t) \; e^{\, i \int_0^T dt \, [\frac{1}{2}\bar{m} |\dot{\vec{x}}|^2 - V(x)]}$$

to represent $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$

if we want $H = -\nabla^2 + \partial_0^2 + m^2$

we set $2\bar{m} = 1 \qquad \bar{m}/2 = \frac{1}{4} \qquad V(x) = m^2$

$$= \int \mathcal{D}X(t) \; e^{\, i \int_0^T dt \, [\frac{1}{4}(-\dot{x}^2) - m^2]}$$

where $-\dot{x}^2 = (\dot{\vec{x}})^2 - (\dot{x}^0)^2$ and $\cdot = \frac{d}{dT}$

If we rescale time $t = t'/2 \qquad \frac{dx^\mu}{dt} = 2\frac{dx^\mu}{dt'}$

$$= \int \mathcal{D}X(t) \; e^{\, i \int_0^{2T} dt' \; \frac{1}{2}(-\dot{x}^2 - m^2)}$$

where now $\dot{x} = \frac{d}{dt'} x^\mu$

then (dropping primes)

$$D_F(x,y) = \int_0^\infty dT \int_{\substack{X(T)=x^\mu \\ X(0)=y^\mu}} \mathcal{D}X \; e^{i \int_0^{2T} dt \; \frac{1}{2}(-\dot{x}^2 - m^2)}$$

b.) Now add a background electromagnetic field. We need to replace the ~~functional~~ integral above by one that solves

$$i \frac{\partial}{\partial T} D(x,y,T) = (D^2 + m^2) \; D(x,y,T)$$

I claim that this is done by adding a Wilson line.

$$D(x,y,T) = \int \mathcal{D}X \; e^{i \int_0^T dt \; [\frac{1}{4}(-\dot{x}^2) - m^2]} \; e^{-ie \int_0^T dt \frac{dx}{dt} \cdot A(x)}$$

To show this, discretize

$$D(x,y,T+\varepsilon) = \int_{-\infty}^{\infty} \frac{dx'}{C} \; e^{i \frac{1}{4} [-\frac{(x-x')^2}{\epsilon}]} \; e^{-i\epsilon m^2} \; e^{ie (x-x')^\mu \cdot A_\mu(\frac{x+x'}{2})}$$

$$D(x',y,T)$$

Expanding to $\mathcal{O}(\epsilon)$ with $(x-x') \sim \sqrt{\epsilon}$

$$D(x,y,T) + \epsilon \frac{\partial}{\partial T} D(x,y,T) = \int \frac{dx'}{C} \, e^{-i \frac{1}{4\epsilon}(x-x')^2}$$

$$\cdot \left(1 - i\epsilon m^2\right)$$

$$\cdot \Big(1 - ie\,(x-x')^\mu A_\mu(x) - \frac{e^2}{2} (x-x')^\mu A_\mu(x) \, (x-x')^\nu A_\nu(x)$$
$$-ie\,(x-x')^\mu \cdot \tfrac{1}{2}(x'-x)^\nu \partial_\nu A_\mu(x) \quad + \cdots \Big)$$

$$\Big(D(x,y,T) + (x'-x)^\nu \partial_\nu D(x,y,T) + \tfrac{1}{2}(x'-x)^\mu (x'-x)^\nu \partial_\mu \partial_\nu D(x,y,T)$$
$$+ \cdots \Big)$$

Now $$\int \frac{dx'}{C} \, e^{-i \frac{1}{4\epsilon}(x-x')^2} = \frac{1}{C} \left[\frac{4\epsilon}{[redacted]\, i} \right]^{4/2}$$

set C so that this is $= 1$

then $$\langle (x-x')^\mu (x-x')^\nu \rangle = g^{\mu\nu} \left(\frac{2\epsilon}{i} \right)$$

Using these relations, the above integral becomes

$\longrightarrow$

$$D(x,y,T) + \epsilon \frac{\partial}{\partial T} D(x,y,T)$$

$$= D(x,y,T) - i\epsilon m^2 D(x,y,T)$$
$$+ \frac{1}{2}(-i2\epsilon) g^{\mu\nu} \partial_\mu \partial_\nu D(x,y,T)$$
$$- \frac{e^2}{2}(-2i\epsilon) g^{\mu\nu} A_\mu A_\nu D(x,y,T)$$
$$- i\frac{e}{2}(+2i\epsilon) g^{\mu\nu} \partial_\nu A_\mu(x) D(x,y,T)$$
$$-ie (+2i\epsilon) g^{\mu\nu} A_\mu \partial_\nu D(x,y,T)$$

$$= D(x,y,T) - i\epsilon\{ m^2 D(x,y,T) + \partial_\mu \partial^\mu D(x,y,T)$$
$$- e^2 A_\mu A^\mu D(x,y,T) + ie(\partial_\mu A^\mu) D(x,y,T)$$
$$+ ie A_\mu \partial^\mu D(x,y,T) \}$$

$$= D(x,y,T) - i\epsilon \{ (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) D(x,y,T)$$
$$+ m^2 D(x,y,T)$$

so indeed.

$$i\frac{\partial}{\partial T} D(x,y,T) = (D^2 + m^2) D(x,y,T)$$

c.) In a non-Abelian gauge theory, this calculation can be repeated as long as the factors of $A_\mu(x) \to A_\mu^a(x)\, t^a$ are <u>to the left</u> of all other elements. Then, in the non-Abelian case:

$$D(x,y,T) = \int \mathcal{D}x \; e^{i\int_0^T dt\, \left(\frac{1}{4}(-\dot{x}^2) - m^2\right)} \cdot \underset{\uparrow}{P}\left(e^{+ig\int dt\, \dot{x}^\mu A_\mu^a t^a}\right)$$

path ordering

yields a solution of

$$i\frac{\partial}{\partial T} D(x,y,T) = \left[\left(\partial_\mu - igA_\mu^a t^a\right)^2 + m^2\right] D(x,y,T)$$