Physics 331 – Problem Set #1

Solutions

1.) The t^a as given satisfy $\quad \mathrm{tr}\,[t^a t^b] = \frac{1}{2}\delta^{ab}$

(note $\mathrm{tr}(t^8)^2 = \frac{1}{12}[1+1+4] = \frac{1}{2}$).

Then

$$\mathrm{tr}\; t^a[t^b, t^c] = \mathrm{tr}\; t^a\, i f^{bcd} t^d = \frac{i}{2} f^{bca}$$

but $= \mathrm{tr}\,[t^a t^b t^c - t^a t^c t^b]$ which is totally antisymmetric by the cyclic invariance of the trace.

So we need only compute one of each triple e.g. $f^{123}, f^{231}, f^{312}$

$$[t^1, t^2] = \left(\tfrac{1}{2}\right)^2 \left[\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & -i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)\right] = \frac{1}{4}\left(\begin{array}{cc|c} 2i & & \\ & -2i & \\ \hline & & 0 \end{array}\right) = \frac{i}{2}\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array}\right)$$

$$[t^1, t^3] = \left(\tfrac{1}{2}\right)^2 \left[\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & \\ \hline & & \end{array}\right)\right] = \frac{1}{4}\left(\begin{array}{cc|c} 0 & -2 & \\ 2 & 0 & \\ \hline & & \end{array}\right) = -\frac{i}{2}\left(\begin{array}{cc|c} 0 & -i & \\ i & 0 & \\ \hline & & 0 \end{array}\right)$$

so $\quad [t^1, t^2] = i t^3 \qquad [t^1, t^3] = -i t^2$

or: $\quad f^{123} = f^{312} = f^{231} = 1 = -f^{132} = -f^{321} = -f^{213}$

$$[t^1,t^4] = \frac{1}{4}\left[\begin{pmatrix}0&1&\\1&0&\\&&\end{pmatrix},\begin{pmatrix}&&1\\&&0\\1&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&0\\&&1\\0&-1&\end{pmatrix} = \frac{i}{2}t^7$$

$$[t^1,t^5] = \frac{1}{4}\left[\begin{pmatrix}0&1&\\1&0&\\&&\end{pmatrix},\begin{pmatrix}&&-i\\&&0\\i&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&\\&&-i\\&-i&\end{pmatrix} = \frac{-i}{2}t^6$$

$$[t^1,t^6] = \frac{1}{4}\left[\begin{pmatrix}0&1&\\1&0&\\&&\end{pmatrix},\begin{pmatrix}&&0\\&&1\\0&1&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&1\\&&0\\-1&0&\end{pmatrix} = \frac{i}{2}t^5$$

$$[t^1,t^7] = \frac{1}{4}\left[\begin{pmatrix}0&1&\\1&0&\\&&\end{pmatrix},\begin{pmatrix}&&0\\&&-i\\0&i&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&-i\\&&0\\-i&0&\end{pmatrix} = \frac{-i}{2}t^4$$

$$[t^2,t^4] = \frac{1}{4}\left[\begin{pmatrix}0&-i&\\i&0&\\&&\end{pmatrix},\begin{pmatrix}&&1\\&&0\\1&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&i\\&&\\i&&\end{pmatrix} = \frac{i}{2}t^6$$

$$[t^2,t^5] = \frac{1}{4}\left[\begin{pmatrix}&-i&\\i&&\\&&\end{pmatrix},\begin{pmatrix}&&-i\\&&0\\i&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&\\&&1\\&-1&\end{pmatrix} = \frac{i}{2}t^7$$

$$[t^2,t^6] = \frac{1}{4}\left[\begin{pmatrix}&-i&\\i&&\\&&\end{pmatrix},\begin{pmatrix}&&0\\&&1\\0&1&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&-i\\&&\\-i&&\end{pmatrix} = \frac{-i}{2}t^4$$

$$[t^2,t^7] = \frac{1}{4}\left[\begin{pmatrix}&-i&\\i&&\\&&\end{pmatrix},\begin{pmatrix}&&\\&&-i\\&i&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&-1\\&&\\1&&\end{pmatrix} = \frac{-i}{2}t^5$$

$$[t^3,t^4] = \frac{1}{4}\left[\begin{pmatrix}1&&\\&-1&\\&&\end{pmatrix},\begin{pmatrix}&&1\\&&0\\1&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&1\\&&\\-1&&\end{pmatrix} = \frac{i}{2}t^5$$

$$[t^3,t^5] = \frac{1}{4}\left[\begin{pmatrix}1&&\\&-1&\\&&\end{pmatrix},\begin{pmatrix}&&-i\\&&0\\i&0&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&-i\\&&\\-i&&\end{pmatrix} = \frac{-i}{2}t^4$$

$$[t^3,t^6] = \frac{1}{4}\left[\begin{pmatrix}1&&\\&-1&\\&&\end{pmatrix},\begin{pmatrix}&&0\\&&1\\0&1&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&0\\&&-1\\0&1&\end{pmatrix} = -\frac{i}{2}t^7$$

$$[t^3,t^7] = \frac{1}{4}\left[\begin{pmatrix}1&&\\&-1&\\&&\end{pmatrix},\begin{pmatrix}&&\\&&-i\\&i&\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}&&\\&&i\\&i&\end{pmatrix} = \frac{i}{2}t^6$$

in this way, the nonzero components of f^{abc} are:

$f^{147} = +\frac{1}{2} = -f^{174}$ $\qquad f^{156} = -\frac{1}{2} = -f^{165}$

$f^{246} = \frac{1}{2} = -f^{264}$ $\qquad f^{257} = +\frac{1}{2} = -f^{275}$

and $\quad f^{345} = \frac{1}{2} = -f^{354} \qquad f^{367} = -\frac{1}{2} = -f^{376}$

and cyclic permutations of these formulae.

$$[t^4, t^5] = \frac{1}{4}\left[\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$= i\cdot\left[\frac{1}{4}\begin{pmatrix} 1&&\\&-1&\\&&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1&&\\&1&\\&&-2\end{pmatrix}\right]$$

$$= i\left[\frac{1}{2}t^3 + \frac{\sqrt{3}}{2}t^8\right]$$

this confirms $\quad f^{453} = \frac{1}{2} = f^{345}$ and gives $\quad f^{458} = \frac{\sqrt{3}}{2}$

$$[t^4, t^6] = \frac{1}{4}\left[\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&+1\\0&1&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0\end{pmatrix} = \frac{i}{2}t^2$$

this confirms $\quad f^{462} = f^{246} = \frac{1}{2}$

$$[t^4, t^7] = \frac{1}{4}\left[\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix} 0&i&0\\i&0&0\\0&0&0\end{pmatrix} = +\frac{i}{2}t^{[redacted]}$$

$$[t^5, t^6] = \frac{1}{4}\left[\begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix} 0&-i&0\\-i&0&0\\0&0&0\end{pmatrix} = -\frac{1}{2}t^1$$

$$[t^5, t^7] = \frac{1}{4}\left[\begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0\end{pmatrix} = \frac{i}{2}t^2$$

these confirm:

$f^{47[redacted]} = f^{[redacted]47} = +\frac{1}{2} \qquad f^{561} = f^{156} = -\frac{1}{2}$

$f^{572} = f^{257} = +\frac{1}{2}$

$$[t^6, t^7] = \frac{1}{4}\left[\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} 0&&\\&1&\\&&-1\end{pmatrix}$$

$$= i\left[-\frac{1}{2}t^3 + \frac{\sqrt{3}}{2}t^8\right]$$

confirming $\quad f^{673} = f^{367} = -\frac{1}{2}$ and giving $\quad f^{678} = \frac{\sqrt{3}}{2}$

we could stop here, since cyclic symmetry gives:

$$f^{845} = \frac{\sqrt3}{2} \qquad f^{867} = \frac{\sqrt3}{2}$$

but we could also check these results:

$$[t^8, t^4] = \frac{1}{2\sqrt3}\cdot\frac12 \left[\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}, \begin{pmatrix}&&1\\&&0\\1&0&\end{pmatrix}\right] = \frac14\frac{1}{\sqrt3}\begin{pmatrix}&&1+2\\&&0\\-2-1&0&\end{pmatrix}$$

$$= \frac{1}{\sqrt3\cdot 4}\begin{pmatrix}&&3\\&&0\\-3&0&\end{pmatrix} = \frac{i\sqrt3}{2}\, t^5 \quad ✓$$

$$[t^8, t^6] = \frac{1}{2\sqrt3}\,\frac12 \left[\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}, \begin{pmatrix}&&0\\&&1\\0&1&\end{pmatrix}\right] = \frac14\frac{1}{\sqrt3}\begin{pmatrix}&&0\\&&1+2\\0&-1-2&\end{pmatrix}$$

$$= i\,\frac{\sqrt3}{2}\, t^7$$

In all, the nonzero elements of f^{abc} are

$$f^{123} = 1$$

$$f^{147} = \tfrac12 \qquad f^{156} = -\tfrac12 \qquad f^{246} = \tfrac12 \qquad f^{257} = -\tfrac12$$

$$f^{345} = \tfrac12 \qquad f^{367} = -\tfrac12 \qquad f^{845} = \frac{\sqrt3}{2} \qquad f^{867} = \frac{\sqrt3}{2}$$

and elements related by total antisymmetry.

Now compute $f^{abc} f^{abd}$.

First consider cases $c = d$:

$$c = d = 1: \quad (f^{231})^2 + (f^{321})^2 + (f^{471})^2 + (f^{561})^2 + (f^{741})^2 + (f^{651})^2$$

$$= 2\cdot\left[(f^{231})^2 + (f^{471})^2 + (f^{561})^2\right]$$

$$= 2\cdot\left[1^2 + \left(\tfrac12\right)^2 + \left(\tfrac12\right)^2\right] = 3$$

$c = d = 2 \qquad 2\left[(f^{132})^2 + (f^{462})^2 + (f^{572})^2\right] = 2(1+\frac{1}{4}+\frac{1}{4}) = 3$

$c = d = 3 \qquad 2\left[(f^{123})^2 + (f^{453})^2 + (f^{673})^2\right] = 2(1+\frac{1}{4}+\frac{1}{4}) = 3$

$c = d = 4 \qquad 2\left[(f^{714})^2 + (f^{624})^2 + (f^{534})^2 + (f^{584})^2\right]$

$$= 2\cdot\left[\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{3}{4}\right] = 3$$

$c = d = 5 \qquad 2\left[(f^{615})^2 + (f^{725})^2 + (f^{345})^2 + (f^{485})^2\right] = 3$

$c = d = 6 \qquad 2\left[(f^{156})^2 + (f^{246})^2 + (f^{736})^2 + (f^{756})^2\right] = 3$

$c = d = 7 \qquad 2\left[(f^{147})^2 + (f^{257})^2 + (f^{367})^2 + (f^{867})^2\right] = 3$

$c = d = 8 \qquad 2\left[(f^{458})^2 + (f^{678})^2\right] = 2\cdot(\frac{3}{4}+\frac{3}{4}) = 3$

For $c \neq d$, it is not possible in most cases to find nonzero contributions for $f^{abc}f^{abd}$. For example, for $c=1$ $d=2$, there is no a, b st.

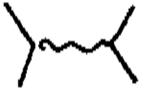
$$f^{ab1}f^{ab2} \text{ is nonzero.}$$

The only exception is $c=3$ $d=8$:

$$f^{ab3}f^{ab8} = 2\left[f^{453}f^{458} + f^{673}f^{678}\right]$$

$$= 2\cdot\left[\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + (-\frac{1}{2})\frac{\sqrt{3}}{2}\right] = 0$$

so indeed $\quad f^{abc}f^{abd} = 3\,\delta^{cd}$ for $SU(3)$

2.) a.)  $= (-ie)^2\, \bar{u}\gamma^\mu u \, \frac{-i}{q^2}\, \bar{u}\gamma_\mu u$

for nonrelativistic fermions, this is. $\bar{u}\gamma^\mu u \approx 2m\, \delta_{ss'}\, \delta^{\mu 0}$

$$\approx +ie^2 \frac{1}{-|\vec{q}|^2}(2m)^2 = -i\left(\frac{+e^2}{|\vec{q}|^2}\right)(2m)^2$$

This implies $\tilde{V}(q) = +\frac{e^2}{|\vec{q}|^2}$ or $V(r) = +\frac{e^2}{4\pi r}$

b.) If $\Psi(x) \to e^{-i\alpha(x) Q}\,\Psi(x)$, we want to find $D_\mu \Psi$ s.t. $D_\mu\Psi \to e^{-i\alpha(x) Q} D_\mu \Psi$ with $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$

For $D_\mu = (\partial_\mu - ieQA_\mu)$:

$$D_\mu \Psi \to (\partial_\mu - ieQA_\mu + iQ\,\partial_\mu \alpha)\, e^{-iQ\alpha(x)}\,\Psi$$

$$= e^{-iQ\alpha(x)}\left[\partial_\mu \cancel{- iQ\,\partial_\mu\alpha(x)} - ieQA_\mu \cancel{+ iQ\,\partial_\mu\alpha(x)}\right]\Psi(x)$$

$$= e^{-iQ\alpha(x)}\left[\partial_\mu - ieQA_\mu\right]\Psi(x)$$

so this form of D_μ satisfies the requirement.

The locally gauge invariant Dirac Lagrangian is then

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + \bar{\Psi} eQ A\!\!\!/ \Psi$$

so $\Delta H = -\,eQ\,\bar{\Psi}\gamma^\mu\Psi A_\mu$

the corresponding Feynman rule is

 $= +ieQ\gamma^\mu$

e.)  $= (+ieQ_1)(+ieQ_2)\,\bar{u}\gamma^\mu u\,\frac{-i}{q^2}\,\bar{u}\gamma_\mu u$

$$= +ie^2 Q_1 Q_2 (2m)^2 \frac{1}{-|\vec{q}|^2} = -i(2m)^2 \frac{e^2 Q_1 Q_2}{|\vec{q}|^2}$$

So $\tilde{V}(q) = +\frac{Q_1 Q_2 e^2}{|\vec{q}|^2}$ or $V(r) = +\frac{Q_1 Q_2 e^2}{4\pi r}$

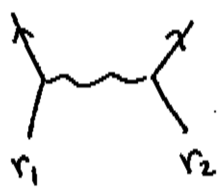
d.) In Yang-Mills theory, the gauge invariant Dirac equation is

$$\mathcal{L} = \bar{\Psi}_r(i\not{D} - m)\psi_r \qquad D_\mu = \partial_\mu - ig A_\mu^a t_r^a$$

for ψ in representation r

$$= \bar{\Psi}_r(i\not{\partial} - m)\psi_r + g A_\mu^a \bar{\Psi}\gamma^\mu t_r^a \psi$$

So  $= ig\gamma^\mu t_r^a$

then  $= (ig)^2\,\bar{u}\gamma^\mu u\,\frac{-i}{q^2}\,\bar{u}\gamma_\mu u \cdot t^a_{r_1} \otimes t^a_{r_2}$

$$= -ig^2(2m)^2 \frac{1}{|\vec{q}|^2}\cdot(t^a_{r_1} \otimes t^a_{r_2})$$

so $$\tilde{V}(q) = + \frac{g^2}{|\vec{q}|^2} \, t^a_{r_1} \otimes t^a_{r_2}$$

a $$V(r) = + \frac{g^2}{4\pi|\vec{r}|} (t^a_{r_1} \otimes t^a_{r_2})$$

e.) Now, in the product representation $r_1 \times r_2$

$$t^a = t^a_{r_1} \otimes 1 + 1 \otimes t^a_{r_2}$$

$$t^2 = (t^a_{r_1} t^a_{r_1}) \otimes 1 + 2 \, t^a_{r_1} \otimes t^a_{r_2} + 1 \otimes t^a_{r_2} t^a_{r_2}$$

$$= C_2(r_1) \cdot \underset{\frown}{1} + 2 \, t^a_{r_1} \otimes t^a_{r_2} + C_2(r_2) \cdot \underset{\frown}{1}$$

The states of the irreducible representation r_I are those eigenstates of t^2 for which $t^2 = C_2(r_I)$

so in the irreducible representation r_I:

$$t^a_{r_1} \otimes t^a_{r_2} = \frac{1}{2} \left[C_2(r_I) - C_2(r_1) - C_2(r_2) \right]$$

In these states

$$V = \frac{g^2}{4\pi r} \cdot \frac{1}{2} \left[C_2(r_I) - C_2(r_1) - C_2(r_2) \right]$$

f) If we take $r_I = \underline{1}$ $\quad C_2(r_I) = 0$. Then

for $r \times \bar{r} \to 1$ $$V = - \frac{g^2 C_2(r)}{4\pi r}$$

There is a nice way to check this: In SU(N)

$$N \times \bar{N} = \underbrace{1}_{\text{singlet}} + \underbrace{(N^2-1)}_{\text{adjoint}}$$

In the singlet state $\quad t^a_N \otimes t^a_{\bar{N}} = \frac{1}{2}\left[-2\, C_2(N)\right] = -\left(\frac{N^2-1}{2N}\right)$

In the adjoint state $\quad t^a_N \otimes t^a_{\bar{N}} = \frac{1}{2}\left[C_2(G) - 2C_2(N)\right]$

$$= \frac{1}{2}\left[N - 2\cdot\frac{N^2-1}{2N}\right] = +\frac{1}{2N}$$

but there are N^2-1 states in G. so

$$\mathrm{tr}\left[\, t^a_N \otimes t^a_{\bar{N}} \,\right] = -\frac{N^2-1}{2N} + (N^2-1)\cdot\frac{1}{2N} = 0$$

This is as required: $\mathrm{tr}\left[t^a_N \otimes t^a_{\bar{N}}\right] = (\mathrm{tr}\, t^a_N)\cdot(\mathrm{tr}\, t^a_{\bar{N}}) = 0\cdot 0 = 0$ ✓

3.) Every non-Abelian Lie group G has an SU(2) subgroup. Let $a,b = 1,2,3$ be the generators of this SU(2). Now consider $\quad \mathrm{tr}\, t^a_r t^b_r = C(r)\, \delta^{ab}$

a.) set $a=b$ and sum over $a=b=1,2,3$

$$\mathrm{tr}\left[\sum_{a=1,2,3} (t^a_r)^2\right] = 3\, C(r)$$

but $\sum_{a=1,2,3} (t_r^a)^2 = J^2$, the quadratic Casimir operator in the SU(2). If r decomposes under SU(2) as

$$r \longrightarrow \sum_i j_i$$

then $\operatorname{tr} J^2 = \sum_i \dim(j_i)\, C_2(j_i)$

$$= \sum_i (2j_i+1)\, j_i(j_i+1)$$

then

$$3\, C(r) = \sum_i (2j_i+1)\, j_i(j_i+1)$$

b.) In SU(N), let a basis vector of the N representation be $\begin{array}{c} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \vdots \\ \vdots \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array}$.

Let $t^1 = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)$ $t^2 = \frac{1}{2}\left(\begin{array}{cc|c} & -i & \\ i & & 0 \\ \hline & 0 & 0 \end{array}\right)$ $t^3 = \frac{1}{2}\left(\begin{array}{cc|c} +1 & 0 & \\ 0 & -1 & 0 \\ \hline & 0 & 0 \end{array}\right)$

t^1, t^2, t^3 generate an SU(2) subgroup of SU(N).

under this subgroup, the N transforms as

1 (spin ½) + (N−2) singlets. (spin 0)

$$3\,C(N) \;=\; \underbrace{2 \cdot \tfrac{3}{4}}_{\text{spin } \frac12} + 0 \;=\; \tfrac{3}{2}$$

so $\quad C(N) \;=\; \tfrac{1}{2}$

the adjoint representation transforms as

$1 \quad$ spin 1 $\qquad \frac12\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \end{array}\right),\; \frac12\left(\begin{array}{cc|c} 0 & -i & \\ i & 0 & \\ \hline & & \end{array}\right),\; \frac12\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & \end{array}\right)$

$2(N-2) \quad$ spin $\frac12 \qquad \left(\begin{array}{c|c} & \alpha\beta \\ \hline & \end{array}\right),\; \left(\begin{array}{c|c} & \\ \hline \alpha\beta & \end{array}\right)$

and singlets

[another way to see this: $\quad N = (\tfrac12) + (N-2)\,(0)$

$\qquad\qquad\qquad\qquad\qquad\quad \bar N = (\tfrac12) + (N-2)\,(0)$

$$N \times \bar N \;=\; (\tfrac12 \times \tfrac12) + (N-2)\cdot \tfrac12 \times 0 + (N-2)\,(0 \times \tfrac12) + (0 \times 0)\text{'s}.$$

$$= \; 1 + 0 \; + \; 2(N-2)\cdot \tfrac12 \; + \; 0\text{'s}.$$

but $\quad N \times \bar N \;=\; (\text{adj}) + 0 \quad$]

then

$$3\,C(G) = \frac{3\cdot 2}{1} + 2(N-2)\cdot 2\cdot \tfrac{3}{4} + 0$$

$$= 3\cdot\left[2 + (N-2)\right] = 3N$$

so $\quad C(G) = N \quad \rightarrow \quad C_2(G) = N$

c.) $\quad (N\times N)_{\text{symmetric}} = (\tfrac{1}{2}\times\tfrac{1}{2})_{\text{symmetric}} + (N-2)(\tfrac{1}{2}\times 0) + 0\text{'s}.$

$$= (1) + (N-2)\cdot(\tfrac{1}{2}) + 0\text{'s}$$

$$(N\times N)_{\text{antisymmetric}} = (\tfrac{1}{2}\times\tfrac{1}{2})_{\text{antisymmetric}} + (N-2)(\tfrac{1}{2}\times 0) + 0\text{'s}.$$

$$= (0) + (N-2)\cdot\tfrac{1}{2}\cdot + 0\text{'s}.$$

so

$$3\,C(\text{anti}) = (N-2)\cdot 2\cdot\frac{3}{4} = 3\cdot\frac{(N-2)}{2}$$

$$3\,C(\text{symm.}) = 3\cdot 2 + (N-2)\cdot 2\cdot\frac{3}{4} = 3\cdot\left(\frac{N+2}{2}\right)$$

now $\quad d(\text{anti}) = \dfrac{N(N-1)}{2} \qquad d(\text{symm}) = \dfrac{N(N+1)}{2}$

$$C_2(\text{anti}) = \frac{d(G)}{d(\text{anti})}\, C(\text{anti}) = \frac{N^2-1}{\frac{N(N-1)}{2}}\;\frac{N-2}{2} = \frac{(N+1)(N-2)}{N}$$

$$C_2(\text{sym}) = \frac{d(G)}{d(\text{sym})}\, C(\text{sym}) = \frac{N^2-1}{\frac{N(N+1)}{2}}\;\frac{N+2}{2} = \frac{(N-1)(N+2)}{N}$$

now $\quad \mathrm{tr}\,(t^2_{r_1 \times r_2}) = [C_2(r_1) + C_2(r_2)]\, d(r_1)\, d(r_2)$

$$= \sum_i C_2(r_i)\, d(r_i) \qquad \text{for } r_1 \times r_2 = \textstyle\sum r_i$$

Check this with $\quad N \times N = (\text{anti}) + (\text{symm.})$

$$[C_2(N) + C_2(N)]\; d(N)\,.\,d(N) = \frac{N^2-1}{2N} \cdot 2 \cdot N^2 = N(N^2-1)$$

$$\sum_i C_2(r_i)\, d(r_i) = C_2(\text{anti}) \cdot d(\text{anti}) + C_2(\text{sym}) \cdot d(\text{sym})$$

$$= \frac{(N+1)(N-2)}{N}\;\frac{N(N-1)}{2} + \frac{(N-1)(N+2)}{N} \cdot \frac{N(N+1)}{2}$$

$$= \frac{(N^2-1) \cdot N}{N} \cdot \left[\frac{(N-2)}{2} + \frac{N+2}{2}\right]$$

$$= N(N^2-1) = (\text{above}) \checkmark$$