Physics 330 – Problem Set 8
Solutions

1.) a.) $\quad S^\dagger S = 1 \qquad S = 1 + iT$

then $\quad (1 - iT^\dagger)(1 + iT) = 1$

$$1 + iT - iT^\dagger + T^\dagger T = 1$$

or $\quad iT + (iT)^\dagger = -T^\dagger T$

Take the matrix element of this equation between $\langle p_1' p_2' |$ and $| p_1 p_2 \rangle$ using

$$\langle p_1' p_2' | iT | p_1 p_2 \rangle = i\mathcal{M}\, (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2)$$

In $T^\dagger T$, insert a complete set of intermediate states $|f\rangle$

Then $\quad (i\mathcal{M} - i\mathcal{M}^*)\,(2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2)$

$$= -\sum_f \mathcal{M}^*(p_1' p_2' \to f)\, \mathcal{M}(p_1 p_2 \to f)\, (2\pi)^4 \delta(p_1' + p_2' - P_f) \times (2\pi)^4 \delta(p_1 + p_2 - P_f)$$

with the sum over all possible final states f

$$-2\, \mathrm{Im}\, \mathcal{M}(p_1 p_2 \to p_1' p_2')\, (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2)$$

$$= -\sum_f \mathcal{M}^*(p_1' p_2' \to f)\, \mathcal{M}(p_1 p_2 \to f)\, (2\pi)^4 \delta(p_1 + p_2 - P_f)\, (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2)$$

Now, for equal mass initial particles, in the CM frame.

$$\sigma_{tot.} = \frac{1}{2E\, 2E\, |v_1 - v_2|} \sum_f (2\pi)^4 \delta^{(4)}(p_1+p_2-P_f) \left| \mathcal{M}(p_1p_2 \to f) \right|^2$$

$$= \frac{1}{4E^2 \cdot 2\frac{P}{E}} \sum_f (2\pi)^4 \delta^{(4)}(p_1+p_2-P_f) \left| \mathcal{M}(p_1p_2 \to f) \right|^2$$

$E_{cm} = 2E$

$p_{cm} = P$

$$= \frac{1}{4E_{cm}\, p_{cm}} \sum_f (2\pi)^4 \delta^{(4)}(p_1+p_2-P_f)$$

$$2E_{cm}\, p_{M} \cdot 2\,\mathrm{Im}\,\mathcal{M} = 4E_{cm}\, p_{cm}\, \mathrm{Im}\,\mathcal{M}$$

so, take $p_1' p_2' \to p_1 p_2$ (forward scattg) in the expression on the previous page.

$$\mathrm{Im}\,\mathcal{M}(p_1p_2 \to p_1p_2) = 2E_{cm}\, p_{cm} \cdot \sigma_{Tot}(p_1p_2)$$

2.) The vertex to be used in this problem is

 $= -i\lambda$

but we will need to be careful about symmetry factors.

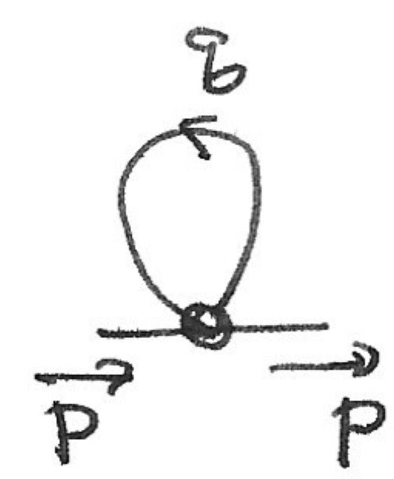
a.) In leading order

$$\mathcal{M}(p_1 p_2 \to p_1' p_2') = -i\lambda$$

$$\sigma = \frac{1}{4 E_{cm} p_{cm}} \frac{1}{16\pi} \int_0^1 d\cos\theta \left(\frac{2 p_{cm}}{E_{cm}}\right) \cdot |\lambda|^2$$

note: ϕ bosons are identical particles, so integrate only over ½ of phase space

$$\sigma = \frac{\lambda^2}{32\pi E_{cm}^2}$$

b.) $-iM^2 =$ 

$$= \frac{\lambda}{2} \underbrace{\int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2}}$$

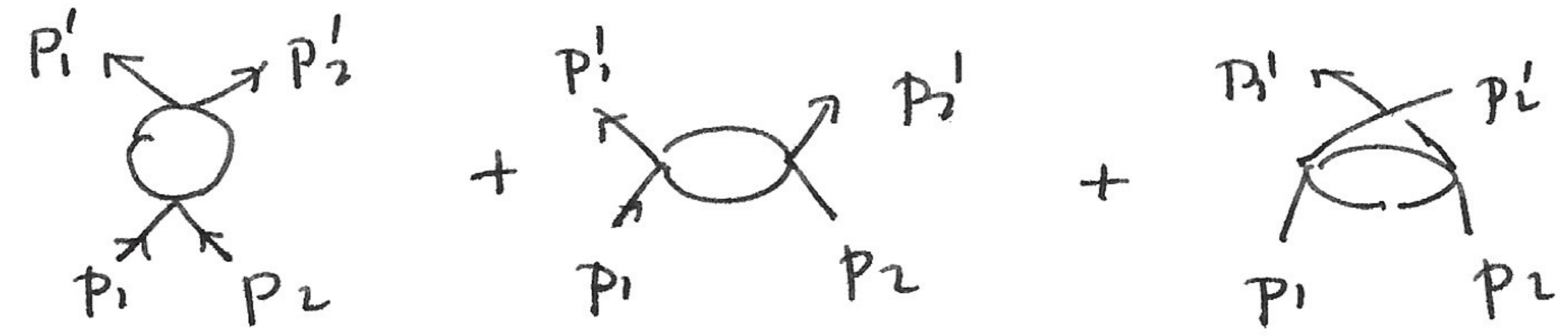
This is a divergent integral but, manifestly, it does not depend on p.

then $\underline{M^2(p^2) = (\text{const})}$

$m^2 = m_0^2 + M^2$ as in class

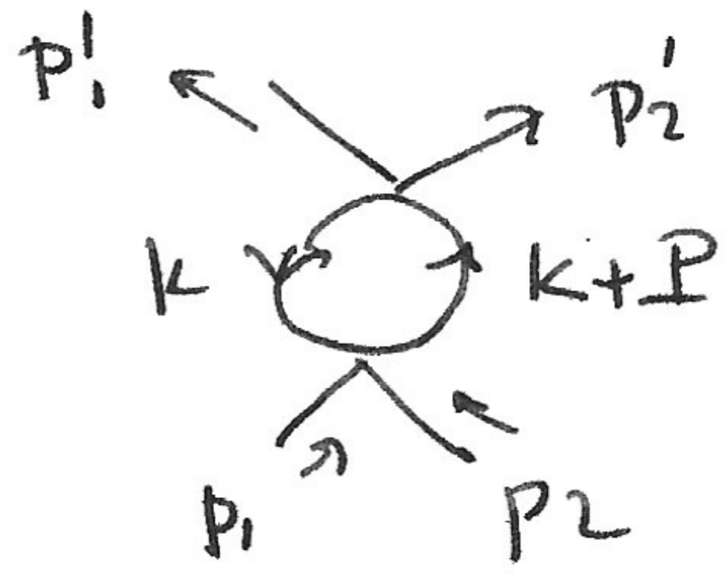
$$\frac{1}{Z} = 1 + \frac{d}{dp^2} M^2 = 1$$

I will compute the remaining diagrams using the physical mass m.

c.) 

give the $O(\lambda^2)$ corrections to $\mathcal{M}$.

Begin with the first diagram: let $P = p_1 + p_2$



$$= \underbrace{(-i\lambda)^2 \cdot \frac{1}{2}}_{\text{symmetry factor}} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+P)^2 - m^2}$$

$$= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[x((k+P)^2 - m_1^2) + (1-x)(k^2 - m_2^2)\right]^2}$$

these masses are equal, but keep them separate for the moment.

$$\text{Denominator} = k^2 + 2x\, k\cdot P + x P^2 - (x m_1^2 + (1-x) m_2^2)$$

$$\mathbb{k} = (k + xP) \qquad \mathbb{k}^2 = k^2 + 2x\, k\cdot P + x^2 P^2$$

$$= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4\mathbb{k}}{(2\pi)^4} \frac{1}{[\mathbb{k}^2 - \Delta]^2}$$

$$\Delta = [x m_1^2 + (1-x) m_2^2 - x(1-x) P^2]$$

$$\frac{1}{Z} = 1 + \frac{d}{dp^2} M^2 = 1$$

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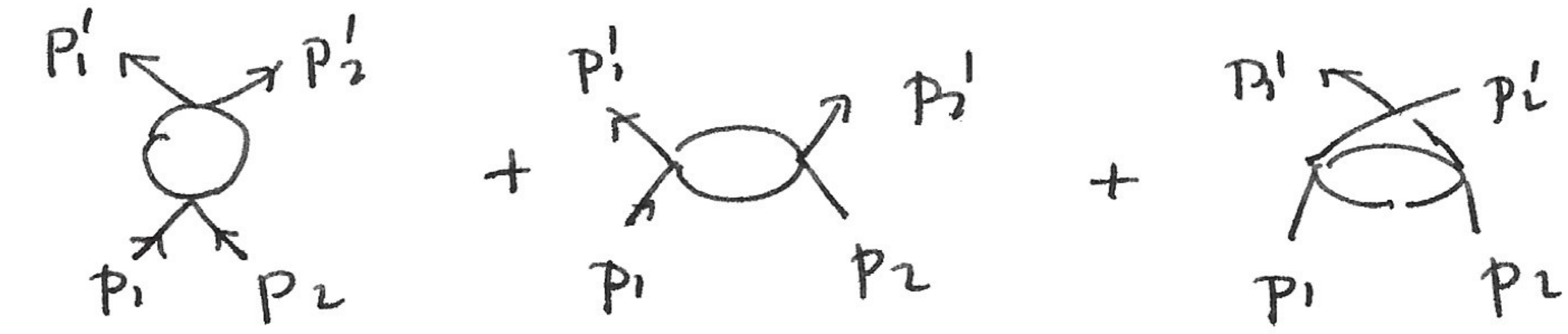

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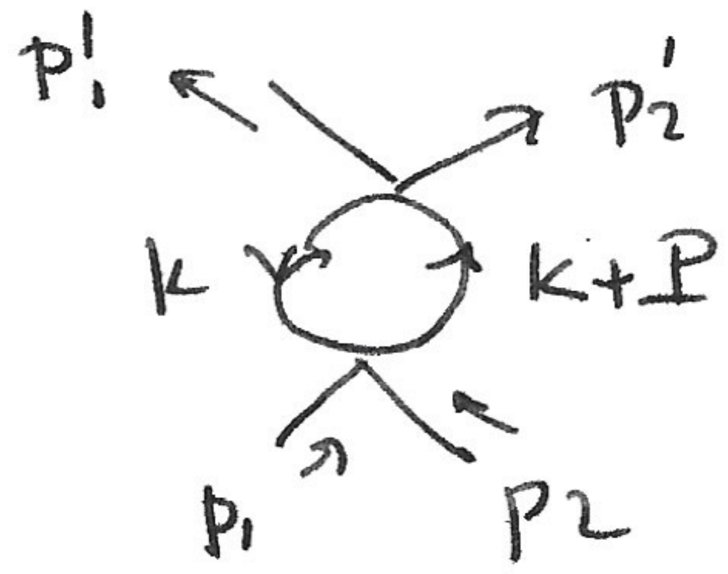

$$= \underbrace{(-i\lambda)^2 \cdot \frac{1}{2}}_{\text{symmetry factor}} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+P)^2 - m^2}$$

$$= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[x((k+P)^2 - m_1^2) + (1-x)(k^2 - m_2^2)\right]^2}$$

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$$\Delta = [x m_1^2 + (1-x) m_2^2 - x(1-x) P^2]$$

Then the integral becomes

$$\int \frac{d^4 \mathbb{k}}{(2\pi)^4} \frac{1}{[\mathbb{k}^2-\Delta]^2} \Bigg|_{\text{regulated}}$$

$$= \frac{i}{(4\pi)^2} \Bigg\{ \log \frac{K^2 + m^2 - x(1-x)P^2}{m^2 - x(1-x)P^2} - 2 \log\left(\frac{K^2 + xm^2 + (1-x)\Lambda^2 - x(1-x)P^2}{xm^2 + (1-x)\Lambda^2 - x(1-x)P^2}\right)$$

$$+ \log \frac{K^2 + \Lambda^2 - x(1-x)P^2}{\Lambda^2 - x(1-x)P^2}$$

$$+ 1 - 1 - 1 + 1$$

$$- \frac{\Delta(P^2, m^2, m^2)}{K^2 + \Delta(P^2, m^2, m^2)} + 2 \frac{\Delta(P^2, m^2, \Lambda^2)}{K^2 + \Delta(P^2, m^2, \Lambda^2)} - \frac{\Delta(P^2, \Lambda^2, \Lambda^2)}{K^2 + \Delta(P^2, \Lambda^2, \Lambda^2)} \Bigg\}$$

Using

$$\log(K^2 + X) = \log K^2 + \frac{X}{K^2} + \cdots$$

we see that the $\log K^2$ terms cancel and all other terms with K^2 vanish as $K^2 \to \infty$. So we can take the limit $K^2 \to \infty$ and ~~find~~

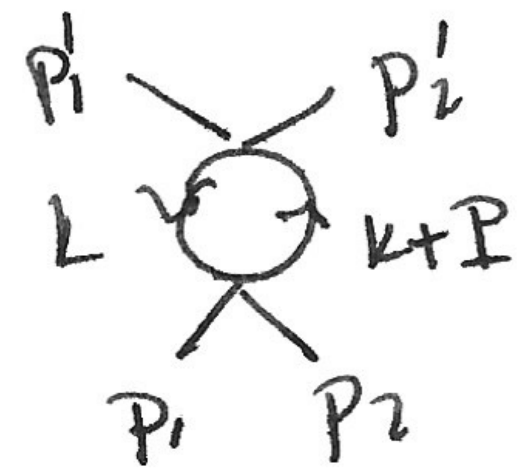
$$\int \frac{d^4 \mathbb{k}}{(2\pi)^4} \frac{1}{[\mathbb{k}^2 - \Delta^2]^2} \Bigg|_{\text{P-V regulated}}$$

$$= \frac{i}{(4\pi)^2} \Bigg\{ \log\left(\frac{1}{m^2 - x(1-x)P^2}\right) - 2 \log \frac{1}{(1-x)\Lambda^2} + \log \frac{1}{\Lambda^2} \Bigg\}$$

$$+ \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)$$

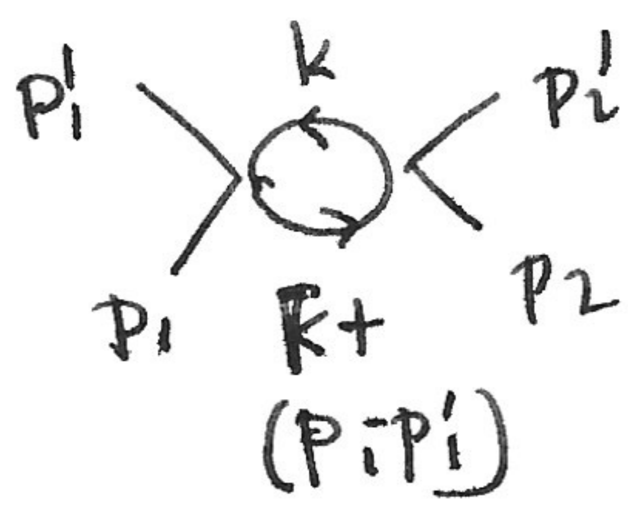
$$= \frac{i}{(4\pi)^2} \log\left(\frac{(1-x)^2 \Lambda^2}{m^2 - x(1-x)P^2}\right) + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)$$

so

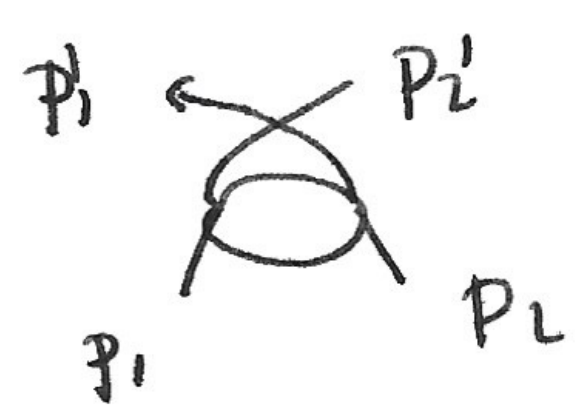


$$= \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx \, \log\left(\frac{[(1-x)^2\Lambda^2]}{[m^2 - x(1-x)P^2]}\right)$$

In this diagram, $P^2 = s$. The other diagrams are crosses of this one



$$= \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx \, \log\left(\frac{(1-x)^2\Lambda^2}{m^2 - x(1-x)t}\right)$$



$$= \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx \, \log\left(\frac{(1-x)^2\Lambda^2}{m^2 - x(1-x)u}\right)$$

then the λ^2 contribution to $i\mathcal{M}$ is

$$(i\mathcal{M})_2 = \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx \, \Big\{ 3\log(1-x)^2 - \log\left(\frac{m^2 - x(1-x)s}{\Lambda^2}\right) - \log\left(\frac{m^2 - x(1-x)t}{\Lambda^2}\right) - \log\left(\frac{m^2 - x(1-x)u}{\Lambda^2}\right)\Big\}$$

$$= \frac{i\lambda^2}{2(4\pi)^2}\Big\{ -6 + 3\log\frac{\Lambda^2}{m^2} - f\left(\frac{s}{m^2}\right) - f\left(\frac{t}{m^2}\right) - f\left(\frac{u}{m^2}\right)\Big\}$$

where $f(z) = \int_0^1 dx \, \log(1 - x(1-x)z)$

d) The value of $(i\mathcal{M})_2$ at threshold,

$$s = (2m)^2 \qquad t = 0 \qquad u = 0$$

is

$$(i\mathcal{M})_2 \text{ at threshold} = \frac{i\lambda^2}{2(4\pi)^2}\Big\{ -6 + 3\log \Lambda^2/m^2 - f\left(\tfrac{4m^2}{m^2}\right) - f(0) - f(0) \Big\}$$

$$f(0) = \int_0^1 dx \, \log(1) = 0$$

Then if $i\mathcal{M}_0$ is the value of the scattering amplitude at threshold

$i\mathcal{M}$ =  +  +  +  $+ \mathcal{O}(\lambda^3)$

$$= i\mathcal{M}_0 + \frac{(-i\lambda^2)}{32\pi^2}\left[f\left(\tfrac{s}{m^2}\right) + f\left(\tfrac{t}{m^2}\right) + f\left(\tfrac{u}{m^2}\right) - f(4) \right]$$

where

$$f(4) = \int_0^1 dx \, \log(1 - 4x(1-x))$$

$$= \int_0^1 dx \, \log((1-2x)^2)$$

$$= 4\int_0^{1/2} dx \, \log(1-2x)$$

$$= 2\int_0^1 dy \, \log(1-y) = -2$$

then

$$i\mathcal{M} = i\Big[\mathcal{M}_0 - \frac{\lambda^2}{32\pi^2}\Big(f\big(\tfrac{s}{m^2}\big) + f\big(\tfrac{t}{m^2}\big) + f\big(\tfrac{u}{m^2}\big) + 2\Big)$$

$$+ \mathcal{O}(\lambda^3)\Big]$$

We should identify λ with $\mathcal{M}_0$ to this order. Then, using the measured value of $\mathcal{M}_0$, the rest of this expression is a well-defined prediction free of divergences.

e.) $$f(z) = \int_0^1 dx \, \log(1 - x(1-x)z)$$

is real for $z < 0$ and, as we will see, for $z < 4$

then $f(\frac{t}{m^2})$, $f(\frac{u}{m^2})$ are real since $t, u < 0$

At threshold, $s = 4m^2$ and above threshold $s/m^2 > 4$

Then the log can have a negative argument in some region of x. Then $f(s/m^2)$ will have an imaginary part.

Let's find the relevant region of x: The argument of the log is

$$1 - x(1-x)\frac{s}{m^2} = \frac{s}{m^2}\left(x^2 - x + m^2/s\right)$$

$$= \frac{s}{m^2}\left(\left(x - \tfrac{1}{2}\right)^2 + \left(m^2/s - \tfrac{1}{4}\right)\right)$$

$$= \frac{s}{m^2}\left(x - \tfrac{1}{2} + \left(\tfrac{1}{4} - m^2/s\right)^{\frac{1}{2}}\right)\left(x - \tfrac{1}{2} - \left(\tfrac{1}{4} - m^2/s\right)^{\frac{1}{2}}\right)$$

so for x in the interval

$$\tfrac{1}{2} - \left(\tfrac{1}{4} - m^2/s\right)^{1/2} < x < \tfrac{1}{2} + \left(\tfrac{1}{4} - m^2/s\right)^{1/2}$$

the log has a negative argument, For s evaluated at $s + i\varepsilon$ the argument $\left(1 - x(1-x)\frac{s}{m^2}\right) = -A - i\varepsilon$

so $\quad \mathrm{Im} \log(\,) = -i\pi$

Then

$$\mathrm{Im}\, f\left(\tfrac{s}{m^2}\right) = \mathrm{Im} \int_0^1 dx \, \log\left(1 - x(1-x)\tfrac{s}{m^2}\right)$$

$$= \int_{\frac{1}{2} - \left(\frac{1}{4} - m^2/s\right)^{1/2}}^{\frac{1}{2} + \left(\frac{1}{4} - m^2/s\right)^{1/2}} dx \quad (-\pi)$$

$$= -2\pi \left(\tfrac{1}{4} - m^2/s\right)^{1/2}$$

$$= -\pi \left(1 - m^2/s\right)^{1/2}$$

$$= -\pi \frac{P}{E} = -\frac{2\pi\, P_{cm}}{E_{cm}}$$

$$\mathrm{Im}\, \mathcal{M} = \left(-\frac{\lambda^2}{32\pi^2}\right)\left(-2\pi \frac{P_{cm}}{E_{cm}}\right)$$

$$= \frac{\lambda^2}{16\pi} \frac{P_{cm}}{E_{cm}}$$

$$= 2 E_{cm} P_{cm} \cdot \frac{\lambda^2}{32\pi E_{cm}^2} = 2 E_{cm} P_{cm}\, \sigma$$

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