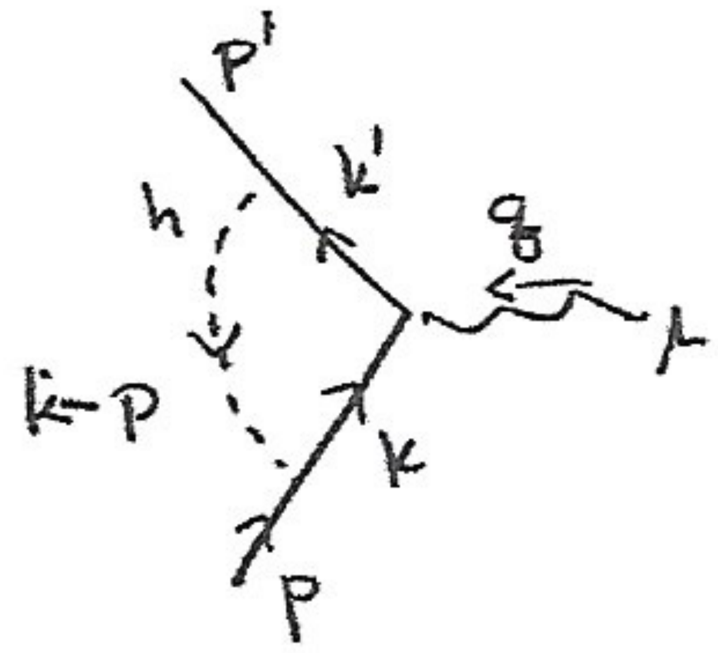
Physics 330 – Problem Set #7

Solutions

1.) a) The hff vertex is  $= -i\frac{y_f}{\sqrt{2}} \cdot 1$

The correction to the electromagnetic vertex due to the Higgs boson is



$$= (-ie)\left(-i\frac{y_f}{\sqrt{2}}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{+i}{(k-p)^2 - m_h^2}$$

$$\cdot\ \bar{u}(p') \cdot 1 \cdot \frac{i(\not{k}'+m)}{k'^2 - m^2}\,\gamma^\mu\,\frac{i(\not{k}+m)}{k^2-m^2} \cdot 1\ u(p)$$

Combine denominators with Feynman parameters

$$= (-ie)\ i \cdot \frac{y_f^2}{2} \int dx\,dy\,dz\ \delta(x+y+z-1) \cdot 2$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\,(\not{k}'+m)\gamma^\mu(\not{k}+m)\,u(p)}{\left[x(k^2-m^2) + y(\underbrace{k'^2}_{k+q}-m^2) + z\left((k-p)^2 - m_h^2\right)\right]^3}$$

simplifying the denominator:

$$\text{Den} = k^2 + y\,2kq + yq^2 + z(-2k\cdot p) + zp^2 - (x+y)m^2 - z\,m_h^2$$

$$= \mathbb{k}^2 - (yq - zp)^2 + yq^2 + zp^2 - (x+y)m^2 - z\,m_h^2$$

$$= \mathbb{k}^2 + y\underbrace{(1-y)}_{x+z}q^2 + z\underbrace{(1-z)}_{x+y}p^2 + 2yz\,p\cdot q - (1-z)m^2 - zm_h^2$$

$$= \mathbb{k}^2 + yz\underbrace{(q^2+p^2+2p\cdot q)}_{p'^2=m^2} + xyq^2 + xz\underbrace{p^2}_{m^2} - (1-z)m^2 - zm_h^2$$

$$= \mathbb{k}^2 + \underbrace{(yz + xz)}_{z(1-z)}m^2 + xyq^2 - (1-z)m^2 - zm_h^2$$

$$= \mathbb{k}^2 + xyq^2 - (1-z)^2m^2 - z\,m_h^2$$

$$= \mathbb{k}^2 - \Delta$$

with $\Delta = (1-z)^2m^2 + z\,m_h^2 - xyq^2$

$$\mathbb{k} = k + yq - zp$$

$$k = \mathbb{k} - yq + zp \qquad k' = \mathbb{k} + (1-y)q + zp$$

the numerator is

$$\bar{u}(p')\,[\,\not{\mathbb{k}} + (1-y)\not{q} + z\not{p} + m]\,\gamma^\mu\,[\,\not{\mathbb{k}} - y\not{q} + z\not{p} + m]\,u(p)$$

$$= \bar{u}(p')\,[\,\not{\mathbb{k}}\gamma^\mu\not{\mathbb{k}} + (\text{linear in } \mathbb{k})$$

$$+ \Big(\underbrace{(1-y)\not{p}'}_{\to m} + \underbrace{[-(1-y)+z]}_{-x}\not{p} + m\Big)\gamma^\mu\Big(-y\not{p}' + \underbrace{(y+z)}_{(1-x)}\underbrace{\not{p}}_{\to m} + m\Big)\Big]u(p)$$

$$= \bar{u}(p') \{ \not{k} \gamma^\mu \not{k} \quad + \quad (\sim k)$$

$$+ \quad [(2-y)m - x\not{p}]\gamma^\mu [(2-x)m - y\not{p}'] \} u(p)$$

under symmetric $\int d^4 k$

$$\not{k}\gamma^\mu \not{k} \to \tfrac{1}{4} k^2 \eta_{\nu\lambda} \gamma^\nu \gamma^\mu \gamma^\lambda = \tfrac{1}{4} k^2 (-2\gamma^\mu) = -\tfrac{1}{2} k^2 \gamma^\mu$$

$$k^\mu \to 0$$

the last line is

$$\bar{u}(p')[(2-x)(2-y)m^2\gamma^\mu - x(2-x)m\overbrace{\not{p}}^{\not{p}'-\not{q}}\gamma^\mu - y(2-y)m\gamma^\mu\not{p}' + xy\,\not{p}\gamma^\mu\overbrace{\not{p}'}^{\not{p}+\not{q}}]u(p)$$

$$\doteq \bar{u}(p')[\, m^2\gamma^\mu[4-2x-2y+xy-(2x-x^2)-(2y-y^2)+xy]$$

$$+ x(2-x)\not{q}\gamma^\mu m - y(2-y)\gamma^\mu\not{q}m$$

$$- xym\not{q}\gamma^\mu + xym\gamma^\mu\not{q} - xy\,\not{q}\gamma^\mu\not{q}\,]\, u(p)$$

under the $\dfrac{d^4k}{[k^2-\Delta]^3}$ integration $\langle x\rangle = \langle y\rangle$ $\langle x^2\rangle = \langle y^2\rangle$ so

$$= \bar{u}(p')[\, m^2\gamma^\mu[4-4(x+y)+2xy-2(x+y)+x^2+y^2]$$

$$+ [x(2-x)+y(2-y)]\tfrac{1}{2}[\not{q},\gamma^\mu]m$$

$$+ xym[\gamma^\mu,\not{q}] - xy\,\not{q}\gamma^\mu\not{q}\,]\, u(p)$$

$$[\gamma^\mu, \not{q}] = -2i\sigma^{\mu\nu}q_\nu$$

$$\not{q}\gamma^\mu\not{q} = 2q^\mu\not{q} - \gamma^\mu q^2 \qquad \bar{u}(p')\not{q}u(p) = 0$$

in all

$$= \bar{u}(p')\Big[[4-6(x+y)+(x+y)^2]\, m\gamma^\mu$$
$$+ \quad x(2-x)+y(2-y)\,(+i)\,\sigma^{\mu\nu}q_\nu \cdot m$$
$$- 2i\,xy\,m\,\sigma^{\mu\nu}q_\nu + xy\,q^2\Big]\,u(p)$$

$$= \bar{u}(p')\Big[4-6(1-z)+(1-z)^2\Big]m\gamma^\mu$$
$$+ \quad (i\sigma^{\mu\nu}q_\nu)m\left[2(1-z)-(x^2+y^2)-2xy\right]$$
$$+ xy\,q^2\Big]\,u(p)$$

$$= \bar{u}(p')\Big[(1+2z+z^2)\,m^2\gamma^\mu + xy\,q^2\gamma^\mu$$
$$+ \left[2(1-z)-(1-z)^2\right](2m^2)\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]\,u(p)$$

$$= \bar{u}(p')\Big[(1+z)^2 m^2\gamma^\mu + xy\,q^2\gamma^\mu$$
$$+ \quad 2m^2(1-z^2)\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]\,u(p)$$

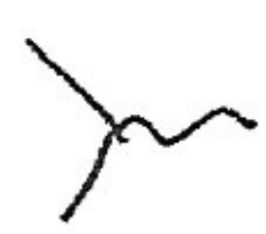
then the full numerator is:

$$\bar{u}(p')\Big[-\tfrac{1}{2}k^2\gamma^\mu + \gamma^\mu\left[(1+z)^2m^2+xy\,q^2\right]$$
$$+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}\,2m^2(1-z^2)\Big]\,u(p)$$

This falls into the form

$$\bar{u}(p')\Big[\gamma^\mu f_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}f_2\Big]\,u(p)$$

c.) In particular,

 +  $= (-ie)\,\bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)\right] u(p)$

with

$$F_2(q^2) = i\, y_f^2 \int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\cdot \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left[k^2 - \Delta + i\varepsilon\right]^3} \cdot \underline{2m^2(1-z^2)}$$

depends on z only

the integral is

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left[k^2-\Delta\right]^3} = \frac{-i}{(4\pi)^2}\,\frac{1}{2}\,\frac{1}{\Delta}$$

then

$$F_2(q^2) = y_f^2 \frac{1}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dx \;\frac{1}{2}\, \frac{2m^2(1-z^2)}{\left[(1-z)^2 m^2 + z m_h^2 - xy\, q^2\right]}$$

$$F_2(0) = \frac{y_f^2}{(4\pi)^2} \int_0^1 dz \frac{m^2(1-z^2)\cdot(1-z)}{\left[(1-z)^2 m^2 + z m_h^2\right]}$$

if $m \gg m_h$

$$F_2(0) = \frac{y_f^2}{(4\pi)^2} \int_0^1 dz \frac{m^2 (1-z)^2 (1+z)}{(1-z)^2 m^2}$$

$$= \frac{y_f^2}{(4\pi)^2} \underbrace{\int dz\,(1+z)}_{3/2} = \frac{3 y_f^2}{32\pi^2}$$

d) For the case $m_h \gg m_f$

$$F_2(0) = \frac{y_f^2}{(4\pi)^2}\int_0^1 dz \, \frac{m_f^2 (1-z)^2(1+z)}{m_f^2(1-z)^2 + m_h^2 z}$$

$$(1-z)^2(1+z) = (1-z^2)(1-z) = 1-z-z^2+z^3$$

$$= \frac{y_f^2}{(4\pi)^2}\int_0^1 dz \, \frac{m_f^2}{m_h^2}\, \frac{1-z-z^2+z^3}{z+(1-z)^2 m_f^2/m_h^2}$$

Notice that if we completely ignore the m_f^2/m_h^2 term in the denominator, the integral is log divergent as $z \to 0$. But this is needed only for the $\underline{1}$ term in the numerator. Also, to evaluate the coefficient of $\underline{1}$, we can set $(1-z)^2 \doteq 1$ in the denominator

$$F_2(0) = \frac{y_f^2}{(4\pi)^2}\frac{m_f^2}{m_h^2}\int_0^1 dz \left[\frac{1}{(z+m_f^2/m_h^2)} + \frac{(-z-z^2+z^3)}{z}\right]\left(1+\mathcal{O}\left(\frac{m_f^2}{m_h^2}\right)\right)$$

$$= \frac{y_f^2}{(4\pi)^2}\frac{m_f^2}{m_h^2}\left[\log\left(\frac{m_h^2}{m_f^2}\right) + \left(-1-\frac{1}{2}+\frac{1}{3}\right)\right]$$

then $$F_2(0) = \frac{y_f^2}{16\pi^2}\frac{m_f^2}{m_h^2}\left(\log\frac{m_h^2}{m_f^2} - \frac{7}{6}\right)$$

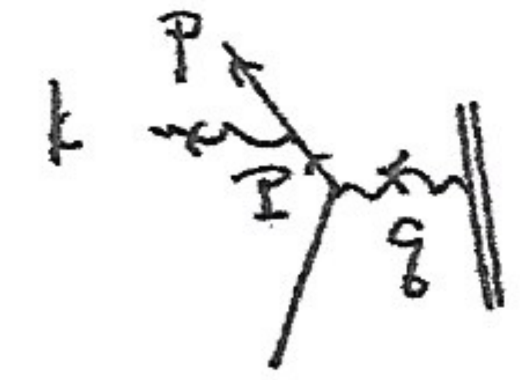
$$= \frac{m_f^4}{8\pi^2 v^2 m_h^2}\left(\log\frac{m_h^2}{m_f^2} - \frac{7}{6}\right)$$

e.) For $m_f = 1.777$ GeV, $v = 246$ GeV, $m_h = 125$ GeV

$$a_\tau^{\text{Higgs}} = F_2(0) = 9.8\times 10^{-10}$$

[thanks to Y. Qian]

2.) a.)



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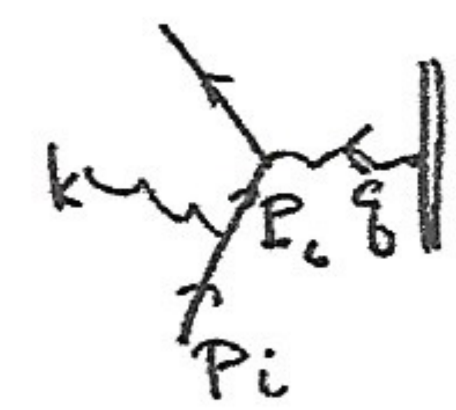

b.) The denominator in the first diagram is

$$P^2 - m_e^2 = (p+k)^2 - m_e^2 = 2p\cdot k$$

For p and k almost parallel, with θ the angle between them

$$2p\cdot k = 2\,E_p E_k - pk\cos\theta$$

If the electron and photon are on shell $E_k = k \quad E_p = \sqrt{p^2+m_e^2}$

for $E_p \gg m_e \qquad E_p = p + \dfrac{m_e^2}{2p} + \dots$

Then $\qquad P^2 = 2pk(1-\cos\theta) + \dfrac{m_e^2}{p}\,k$

$$\cong pk\left(\theta^2 + \frac{m_e^2}{p^2}\right) \approx pk\,\theta^2 \quad \text{for } m_e^2 \to 0$$

The phase space integral is $\quad d\cos\theta = d\theta\,\sin\theta \approx d\theta\cdot\theta$

so the phase space integral will be

$$\int_0 d\theta\cdot\theta\,\frac{1}{\left[\theta^2 + m_e^2/p^2\right]^2}$$

In the calculation below, there is another θ^2 in the numerator from the matrix element. But still the integral is

$$\int_0 d\theta\cdot\theta\,\frac{\theta^2}{\left[\theta^2+m_e^2/p^2\right]^2} \qquad \text{which still} \to \infty \quad \text{as } m_e/p \to 0$$

In the other diagram, the denominator is

$$\frac{1}{P_i^2 - \hat{m}_e^2} = \frac{1}{(p_i - k)^2 - m_e^2} = \frac{1}{-2p_i \cdot k}$$

which has a similar singularity when p_i and k are almost parallel.

c.) For final state radiation, $\vec{P} \parallel \hat{z}$, and $p\,k$ in the $\hat{x}-\hat{z}$ plane, and $E_p, E_k \gg m_e$, we can write ($m_e = 0$ from here on)

$$p = \left(E_p,\ p_\perp,\ 0,\ E_p - \frac{p_\perp^2}{2E_p}\right) \qquad \text{so} \qquad p^2 = 0$$

$$k = \left(E_k,\ -p_\perp,\ 0,\ E_k - \frac{p_\perp^2}{2E_k}\right)$$

$$P = \left(E,\ 0\ \ 0,\ E - \frac{p_\perp^2}{2E_p} - \frac{p_\perp^2}{2E_k}\right)$$

Let $\quad E_p = (1-z)E \qquad E_k = zE \qquad 0 < z < 1$

then
$$P^2 \cong E^2 - \left(E - \frac{p_\perp^2}{2E(1-z)} - \frac{p_\perp^2}{2Ez}\right)^2$$

$$\cong E^2 - \left(E - \frac{p_\perp^2}{2E}\,\frac{z + (1-z)}{z(1-z)}\right)^2$$

$$P^2 \cong \frac{p_\perp^2}{[z(1-z)]}$$

Write the corresponding polarization vectors:

$$u(P) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u(p) = \sqrt{2(1-z)E}\begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}$$

where the rotated spinor is $\xi = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = \begin{pmatrix} -\frac{P_\perp}{2E(1-z)} \\ 1 \end{pmatrix} + O(P_\perp^2)$

The photon polarization vectors of definite helicity are

$$\varepsilon_R^{\mu *} = \frac{1}{\sqrt{2}}(0\ \ \cos\theta_k, -i, -\sin\theta_k)$$

or

$$\varepsilon_R^{\mu *} = \frac{1}{\sqrt{2}}\left(0, 1, -i, \frac{p_\perp}{Ez}\right)$$

$$\varepsilon_L^{\mu *} = \frac{1}{\sqrt{2}}\left(0, 1, +i, \frac{p_\perp}{Ez}\right)$$

e) Write the fermion propagator as $\frac{i\cancel{P}}{P^2} \cong \frac{i\sum_s u^{(s)}(P)\bar{u}^{(s)}(P)}{P^2}$

Then the diagram.

$$\cong (-ie)^2\ \bar{u}(p)\,\gamma\cdot\varepsilon^*(k)\ i\,\frac{u^{(s)}(P)\,\bar{u}^{(s)}(P)}{P^2}$$

$$\cdot\ \gamma^\mu u(P_i)$$

$$\cong [-ie\ \ \bar{u}(p)\,\gamma\cdot\varepsilon^*(k)\,]\frac{u(P)}{P^2}\ \cdot\ e\ \ \bar{u}(P)\,\gamma^\mu u(P_i)$$

$$\cong [e\ \ \bar{u}(p)\,\gamma\cdot\varepsilon^*(k)\,u(P)\,]\frac{1}{P^2}\ \cdot \begin{pmatrix} \text{0th order} \\ \text{matrix element} \\ \text{for } P_i \to P \\ \text{scattering} \end{pmatrix}$$

f) For emission of γ_R :

$$e\ \bar{u}(p)\ \gamma \cdot \varepsilon^*_R\ u(P)$$

$$= e\sqrt{2E(1-z)}\ \left(-\frac{p_\perp}{2E(1-z)},\ 1 \,\middle|\, 0, 0\right) \overset{\gamma^0}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\cdot \begin{pmatrix} 0 & (-\vec{\sigma}\cdot\vec{\varepsilon}^*) \\ \vec{\sigma}\cdot\vec{\varepsilon}^* & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sqrt{2E}$$

$$= e\ 2E\sqrt{1-z}\ \left(-\frac{p_\perp}{2E(1-z)},\ 1\right) \frac{1}{\sqrt{2}} \left(\sigma^1 - i\sigma^2 + \frac{p_\perp}{Ez}\sigma^3\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e\ 2E\sqrt{1-z}\ \left(-\frac{p_\perp}{2E(1-z)},\ 1\right) \frac{1}{\sqrt{2}} \begin{pmatrix} p_\perp/Ez & 0 \\ 2 & -p_\perp/Ez \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e\ 2E\sqrt{1-z}\ \frac{1}{\sqrt{2}} \left(-\frac{p_\perp}{Ez}\right) + \mathcal{O}(p_\perp^2)$$

$$= -e\frac{\sqrt{2(1-z)}}{z}\ p_\perp$$

for emission of γ_L

$$e\ \bar{u}(p)\ \gamma \cdot \varepsilon^*_L\ u(P)$$

$$= e\sqrt{2E(1-z)}\ \left(-\frac{p_\perp}{2E(1-z)},\ 1\right)\ \vec{\sigma}\cdot\vec{\varepsilon}^*_L \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{2E}$$

$$= e\ \frac{2E\sqrt{(1-z)}}{\sqrt{2}} \left(-\frac{p_\perp}{2E(1-z)},\ 1\right) \begin{pmatrix} p_\perp/Ez & 2 \\ 0 & -p_\perp/Ez \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e\sqrt{2}\sqrt{1-z}\ E\left[-\frac{p_\perp}{Ez} - \frac{p_\perp}{E(1-z)}\right]$$

again:

$$e\,\bar{u}(p)\,\gamma\cdot\varepsilon_R^*\,u(P) = -e\sqrt{2(1-z)}\;p_\perp\,\frac{1}{z}$$

$$e\,\bar{u}(p)\,\gamma\cdot\varepsilon_L^*\,u(P) = -e\sqrt{2(1-z)}\;p_\perp\,\frac{1}{z(1-z)}$$

g.) Let $i\mathcal{M}_0$ be the 0th order scatt'g amplitude

$$i\mathcal{M} = -e\sqrt{2(1-z)}\;p_\perp\cdot\begin{Bmatrix}\frac{1}{z} & (\gamma_R)\\ \frac{1}{z(1-z)} & (\gamma_L)\end{Bmatrix}\cdot\frac{1}{P^2}\cdot(i\mathcal{M}_0)$$

the phase space integral for $e^-(P_i) + X \to e^-(p) + \gamma(k) + Y$

$$\int\frac{d^3P_Y\,d^3p\,d^3k}{(2\pi)^3 2E_Y\,(2\pi)^3 2E_p\,(2\pi)^3 2E_k}\,(2\pi)^4\delta^{(4)}(P_i + P_X - P_Y + p + k)$$

we can write

$$d^3p\,d^3k = d^3P\,d^3k \qquad E_k = k = Ez$$

$$2E_p = 2E_P(1-z) \qquad dk^3 = E\,dz$$

$$\delta^{(4)}(P_i + P_X - (P_Y + p + k)) = \delta^{(4)}(P_i + P_X - P_Y + P)$$

then the cross section formula factorizes into

$$d\sigma = d\sigma\Big|_{0\text{th}}\cdot\int\frac{d^3k}{(2\pi)^3 2k\,(1-z)}\cdot\frac{e^2\,2(1-z)\,p_\perp^2}{(P^2)^2}$$

$$\cdot\begin{Bmatrix}\frac{1}{z^2} & (\gamma_R)\\ \frac{1}{z^2(1-z)^2} & (\gamma_L)\end{Bmatrix}$$

$$d\sigma = d\sigma_0 \cdot \int \frac{dz\, d^2p_\perp}{(2\pi)^3\, 2z(1-z)} \frac{e^2\, 2(1-z)\, p_\perp^2}{\left(\frac{(p_\perp^2)^2}{z^2(1-z)^2}\right)} \cdot \begin{cases} \frac{1}{z^2} \\ \frac{1}{z^2(1-z)^2} \end{cases}$$

$$= d\sigma_0 \int \frac{dz\, \pi\, d(p_\perp^2)}{8\pi^3 \cdot 2z(1-z)} \frac{e^2\, 2(1-z)\, z^2(1-z)^2\, p_\perp^2}{(p_\perp^2)^2}$$

$$\cdot \left[\underset{\substack{\uparrow \\ \text{emission of} \\ \gamma_R}}{\frac{1}{z^2}} + \underset{\substack{\uparrow \\ \text{emission of} \\ \gamma_L}}{\frac{1}{z^2(1-z)^2}} \right]$$

$$= d\sigma_0 \int \frac{dz\, d(p_\perp^2)}{8\pi^2}\, e^2\, \frac{1}{z}\, \frac{z^2(1-z)^2}{z^2(1-z)^2} \left(1+(1-z)^2\right) \frac{p_\perp^2}{(p_\perp^2)^2}$$

$$d\sigma = d\sigma_0 \left(\frac{\alpha}{2\pi}\right) \int \frac{dz}{z} \int \frac{dp_\perp^2}{p_\perp^2} \left(1+(1-z)^2\right)$$