Physics 330 – Problem Set #6

Solutions

1.) a.) in the CM frame, or, the rest frame of the decaying particle, $Q^\mu = (Q, \vec{0})$. So

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

Then $\vec{p}_3 = -(\vec{p}_1 + \vec{p}_2)$ must lie in the $(\vec{p}_1, \vec{p}_2)$ plane

b) $$x_1 = \frac{2p_1 \cdot Q}{Q^2}$$

with $Q = (Q, \vec{0})$ $p_1 = (E_1, \vec{p}_1)$

$$x_1 = \frac{2E_1}{Q}$$

$$x_1 + x_2 + x_3 = \frac{2}{Q}(E_1 + E_2 + E_3) = \frac{2Q}{Q} = 2$$

$$E_1 = \frac{Q}{2}x_1 \qquad E_2 = \frac{Q}{2}x_2 \qquad E_3 = \frac{Q}{2}x_3$$

$$|\vec{p}_1| = \left[\left(\frac{Q}{2}\right)^2 x_1^2 - m_1^2\right]^{1/2} = \frac{Q}{2}\left(x_1^2 - \frac{4m_1^2}{Q^2}\right)^{1/2}$$

and similarly

$$p_2 = \frac{Q}{2}\left(x_2^2 - \frac{4m_2^2}{Q^2}\right)^{1/2} \qquad p_3 = \frac{Q}{2}\left(x_3^2 - \frac{4m_3^2}{Q^2}\right)^{1/2}$$

write $y_1 = \left(x_1^2 - \frac{4m_1^2}{Q^2}\right)^{1/2}$ etc

c.) $$(p_1+p_2)^2 = (Q-p_3)^2$$
$$= Q^2 - 2Q\cdot p_3 + m_3^2$$
$$= Q^2(1-x_3) + m_3^2$$

and, similarly,

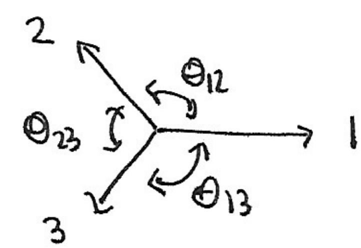
$$(p_1+p_3)^2 = Q^2(1-x_2) + m_2^2 \qquad (p_2+p_3)^2 = Q^2(1-x_1) + m_1^2$$

d.) also $$(p_1+p_2)^2 = m_1^2 + m_2^2 + 2p_1\cdot p_2$$
$$= m_1^2 + m_2^2 + 2(E_1E_2 - p_1p_2\cos\theta_{12})$$
$$= m_1^2 + m_2^2 + \frac{Q^2}{2}(x_1x_2 - y_1y_2\cos\theta_{12})$$

setting this equal to the above:

$$\cos\theta_{12} = \frac{\frac{Q^2}{2}x_1x_2 - Q^2(1-x_3) + m_1^2 + m_2^2 - m_3^2}{(Q^2/2)\,y_1y_2}$$

the right-hand side is determined by the x_i. So $\cos\theta_{12}$ (and, similarly $\cos\theta_{13}$, $\cos\theta_{23}$) is determined as a function of the x_i. Then the whole configuration in the event plane



is fixed once the x_i are fixed.

e.) Now evaluate $\int d\Pi_3$

$$\int d\Pi_3 = \int \frac{d^3p_1\, d^3p_2\, d^3p_3}{(2\pi)^3 2E_1 (2\pi)^3 2E_2 (2\pi)^3 2E_3} (2\pi)\, \delta(Q-(E_1+E_2+E_3))\, (2\pi)^3 \delta(\vec{p_1}+\vec{p_2}+\vec{p_3})$$

$$= \int \frac{d^3p_1\, d^3p_2}{(2\pi)^6\, 2E_1\, 2E_2\, 2E_3} (2\pi)\, \delta(Q-(E_1+E_2+E_3))$$

Choose the $\hat{3}$ axis $\parallel$ to $\vec{p}_1$ and $\vec{p}_2$ in the $\hat{3}$-$\hat{1}$ plane

$$\underbrace{\hat{3} \text{ axis } \parallel \text{ to } \vec{p}_1}_{\int d\Omega = 4\pi} \qquad \underbrace{\vec{p}_2 \text{ in the } \hat{3}\text{-}\hat{1}}_{\int d\phi = 2\pi}$$

$$= \int \frac{dp_1\, p_1^2\, dp_2\, p_2^2}{(2\pi)^6\, 2E_1\, 2E_2\, 2E_3}\, 8\pi^2\, d\cos\theta_{12}\; 2\pi\, \delta(Q-E_1-E_2-E_3(\vec{p}_1,\vec{p}_2))$$

$$E_3 = \left[[p_1^2 + 2p_1p_2\cos\theta_{12} + p_2^2] + m_3^2\right]^{1/2}$$

$$\frac{dE_3}{d\cos\theta_{12}} = \frac{1}{2} \cdot \frac{2p_1\, p_2}{E_3}$$

$$= \int \frac{1}{4\pi^3}\, \frac{dp_1\, p_1^2\, dp_2\, p_2^2}{8\, E_1 E_2 E_3}\, \frac{1}{\left(\frac{p_1 p_2}{E_3}\right)}$$

$dp_1\, p_1 = dE_1\, E_1 \qquad dp_2\, p_2 = dE_2\, E_2 \qquad$ since $\quad E_1^2 = p_1^2 + m_1^2$ etc.

$$= \int \frac{1}{32\pi^3}\, \frac{dE_1\, E_1\, dE_2\, E_2}{E_1 E_2}$$

f.) $\quad dE_1 = \frac{Q}{2} dx_1 \qquad dE_2 = \frac{Q}{2} dx_2$

$$= \frac{Q^2}{128\pi^3} \int dx_1\, dx_2$$

again $\quad \int d\Pi_3 = \frac{Q^2}{128\pi^3} \int dx_1\, dx_2$

g.) $m_{23}^2 = Q^2(1-x_1) + m_1^2$ $dm_{23}^2 = Q^2 dx_1$

$m_{13}^2 = Q(1-x_2) + m_2^2$ $dm_{13}^2 = Q^2 dx_2$

$$\int d\Pi_3 = \frac{1}{128\pi^3}\frac{1}{Q^2}\int dm_{23}^2\, dm_{13}^2$$

h.) For $m_1 = m_2 = m_3 = 0$ $E_1 = p_1 = \frac{Q}{2} x_1$ similarly for $x_2\, x_3$

The maximum value of x_1 corresponds to the configuration

$$\overset{2}{\leftarrow}\!\leftarrow \longrightarrow 1$$

$$x_1 = x_2 + x_3 = 1$$

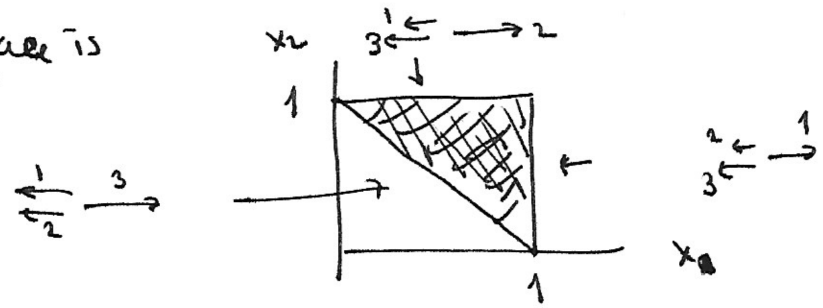
similarly the maximum value of x_2 is $x_2 = 1$

from

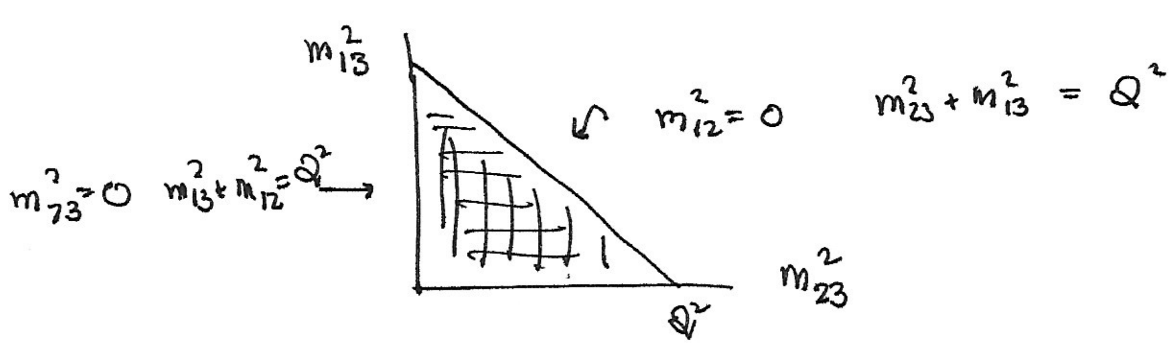
$$\overset{1}{\leftarrow}\!\underset{3}{\leftarrow} \longrightarrow 2$$

also for x_3, the maximum value is $x_3 = 1$ or $x_1 + x_2 = 1$

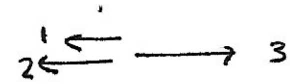
then the region in the (x_1, x_2) plane covered by 3-body phase space is



Translate this to the (m_{23}^2, m_{13}^2) plane $0 < m_{ij}^2 < Q^2$



i.) For $m_1 = m_2 = 0 \quad m_3 > 0$

For 3 recoiling again 1,2 

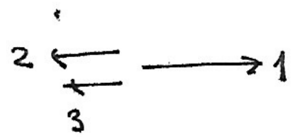
$$P_3 = (P_1 + P_2) = \frac{Q}{2}(x_1 + x_2) = \frac{Q}{2}(2 - x_3)$$

$$\frac{Q}{2}\left(x_3^2 - \frac{4m_3^2}{Q^2}\right)^{1/2} = \frac{Q}{2}(2 - x_3)$$

$$\cancel{x_3^2} - \frac{4m_3^2}{Q^2} = 4 - 4x_3 + \cancel{x_3^2}$$

$$x_3 = \left(1 + m_3^2/Q^2\right) \qquad x_1 + x_2 = \left(1 - \frac{m_3^2}{Q^2}\right)$$

For 1 recoiling agent 2,3



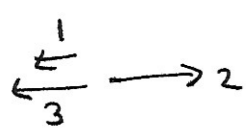
$$x_1 = x_2 + \left(x_3^2 - \frac{4m_3^2}{Q^2}\right)^{1/2}$$

$$(x_1 - x_2)^2 = x_3^2 - \frac{4m_3^2}{Q^2} \qquad x_3 = 2 - (x_1 + x_2)$$

$$\cancel{x_1^2} - 2x_1x_2 + \cancel{x_2^2} = 4 - 4(x_1 + x_2) + \cancel{x_1^2} + 2x_1x_2 + \cancel{x_2^2} - \frac{4m_3^2}{Q^2}$$

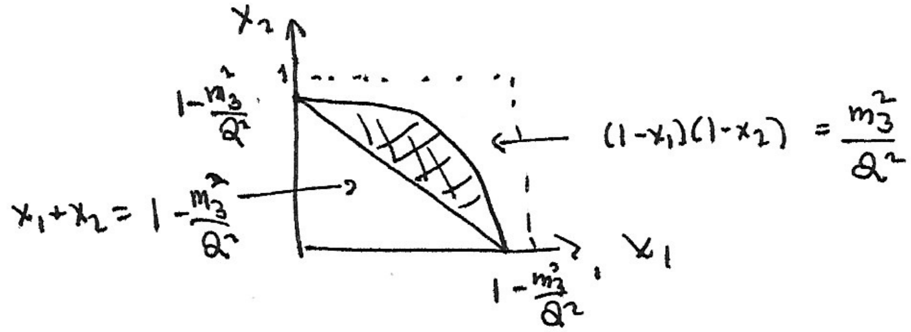
$$\frac{4m_3^2}{Q^2} = 4 - 4(x_1 + x_2) + 4x_1x_2$$

$$\frac{m_3^2}{Q^2} = (1 - x_1)(1 - x_2)$$

 gives the same equation

note! for $x_1 = 0 \quad x_2 = 1 - \frac{m_3^2}{Q^2}$

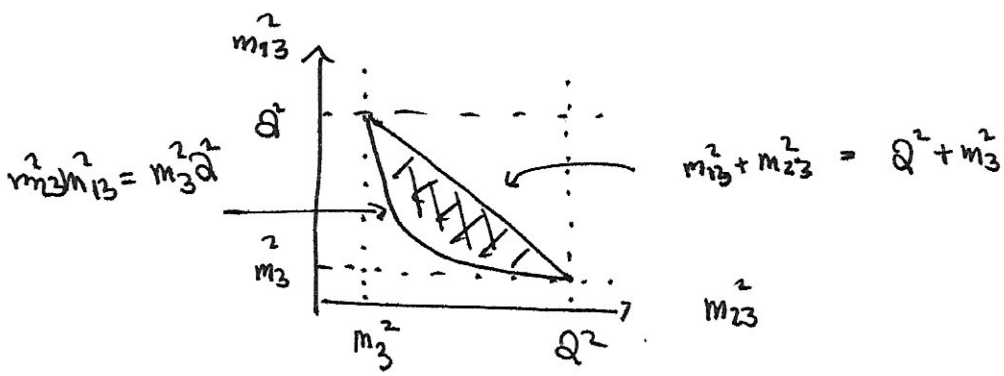
then the region in the $(x_1\, x_2)$ plane for 3-body phase space is



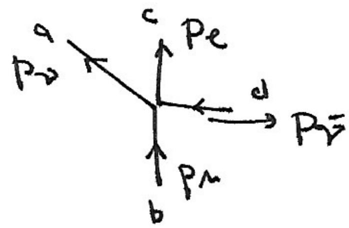
translating into the m_{23}^2 m_{13}^2 plane

$$x_1 = 1-\frac{m_3^2}{Q} \Rightarrow m_{23}^2 = m_3^2$$

$$x_1 = 0 \Rightarrow m_{23}^2 = Q^2$$



2.) a) The ΔH gives a vertex



$$= -i\frac{G_F}{\sqrt{2}}\left(\gamma^\alpha(1-\gamma^5)\right)_{ab}\,\gamma_\alpha(1-\gamma^5)_{cd}$$

then, with $m_e = m_\nu = m_{\bar\nu} = 0$

$$i\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\,\bar{u}(p_\nu)\gamma^\alpha(1-\gamma^5)u(p_\mu)\;\bar{u}(p_e)\gamma_\alpha(1-\gamma^5)v(p_{\bar\nu})$$

b.) Square this and sum over spins; $\left(\left[\bar\Psi\gamma^\mu(1-\gamma^5)\psi\right]^\dagger = \bar\Psi\gamma^\mu(1-\gamma^5)\psi\right)$

$$\sum_{spin}|\mathcal{M}|^2 = \frac{G_F^2}{2}\,\mathrm{tr}\left[\not{p}_\nu\,\gamma^\alpha(1-\gamma^5)(\not{p}_\mu+m_\mu)\gamma^\beta(1-\gamma^5)\right]$$
$$\cdot\,\mathrm{tr}\left[\not{p}_e\gamma_\alpha(1-\gamma^5)\not{p}_{\bar\nu}\,\gamma_\beta(1-\gamma^5)\right]$$

work out the traces, note that γ^5 anticommutes with $\not{p}_\mu$, commutes w. m

$$(1-\gamma^5)\not{p}_\mu\gamma_\beta(1-\gamma^5) = \not{p}\gamma_\beta(1-\gamma^5)(1-\gamma^5) = \not{p}\gamma_\beta\cdot 2(1-\gamma^5)$$
$$(1-\gamma^5)\not{p}_\mu\, m_\mu(1-\gamma^5) = \not{p}_\mu m_\mu(1+\gamma^5)(1-\gamma^5) = 0$$

then

$$\mathrm{tr}\,\not{p}_\nu\gamma^\alpha(1-\gamma^5)(\not{p}_\mu+m_\mu)\gamma^\beta(1-\gamma^5)$$
$$= 2\,\mathrm{tr}\,\not{p}_\nu\gamma^\alpha\not{p}_\mu\gamma^\beta(1-\gamma^5)$$
$$= 2\cdot 4\left[p_\nu^\alpha p_\mu^\beta - p_\nu\cdot p_\mu\,\eta^{\alpha\beta} + p_\nu^\beta p_\mu^\alpha + i\epsilon^{\gamma\alpha\delta\beta}p_{\nu\gamma}p_{\mu\delta}\right]$$
$$= 8\left[p_\nu^\alpha p_\mu^\beta - p_\nu\cdot p_\mu\,\eta^{\alpha\beta} + p_\nu^\beta p_\mu^\alpha - i\epsilon^{\gamma\delta\alpha\beta}p_{\nu\gamma}p_{\mu\delta}\right]$$

similarly $\mathrm{tr}\left[\not{p}_e\gamma_\alpha(1-\gamma^5)\not{p}_{\bar\nu}\gamma_\beta(1-\gamma^5)\right]$

$$= 8\left[p_{e\alpha}p_{\bar\nu\beta}\ [redacted]\ p_e\cdot p_{\bar\nu}\,\eta_{\alpha\beta} + p_{e\beta}p_{\bar\nu\alpha} - i\epsilon^{\gamma\delta}{}_{\alpha\beta}\,p_{e\gamma}p_{\bar\nu\delta}\right]$$

dot these into each other.

(term symmetric in ($\alpha \leftrightarrow \beta$) × term antisymmetric in ($\alpha \leftrightarrow \beta$) give 0)

$$= 64 \cdot \Big[2\, p_\nu \cdot p_e \; p_\mu \cdot p_{\bar\nu} + 2\, p_\nu \cdot p_{\bar\nu} \; p_\mu \cdot p_e - \cancel{4\, p_\nu \cdot p_\mu \; p_e \cdot p_{\bar\nu}}$$

$$+ \cancel{4\, p_\nu \cdot p_\mu \; p_e \cdot p_{\bar\nu}} - \epsilon^{\gamma\delta\alpha\beta} \epsilon^{\eta\theta}{}_{\alpha\beta} \, p_{\nu\gamma} p_{\mu\delta} p_{e\eta} p_{\bar\nu\theta} \Big]$$

In the last term, the sign is crucial; be careful

$$\epsilon^{\gamma\delta\alpha\beta} \epsilon^{\eta\theta}{}_{\alpha\beta} = \;?$$

Do an example!

$$\epsilon^{\gamma\delta 34} \epsilon^{\eta\theta}{}_{34} + \epsilon^{\gamma\delta 43} \epsilon^{\eta\theta}{}_{43} = 2\left(\eta^{\gamma\eta} \eta^{\delta\theta} - \eta^{\gamma\theta} \eta^{\delta\eta}\right)$$

then $\epsilon^{\gamma\delta\alpha\beta} \epsilon^{\eta\theta}{}_{\alpha\beta} = 2\left(p_\nu \cdot p_e \; p_\mu \cdot p_{\bar\nu} - p_\nu \cdot p_{\bar\nu} \; p_\mu \cdot p_e\right)$

$$= 64 \cdot 4 \cdot p_\nu \cdot p_e \; p_\mu \cdot p_{\bar\nu}$$

then

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = 128 \, \frac{G_F^2}{2} \; p_\nu \cdot p_e \; p_\mu \cdot p_{\bar\nu}$$

μ spin avg.

c.) The muon decay rate is then $Q^2 = m_\mu^2$

$$\Gamma = \frac{1}{2m_\mu} \frac{m_\mu^2}{128\pi^3} \int dx_e \, dx_{\bar\nu} \cdot 128 \frac{G_F^2}{2} \; p_\nu \cdot p_e \; p_\mu \cdot p_{\bar\nu}$$

$$= \frac{1}{4\pi^3} m_\mu \int dx_e \, dx_{\bar\nu} \; \frac{m_{\nu e}^2}{2} \; Q \cdot \frac{Q}{2} x_{\bar\nu}$$

and $\quad m_{\nu e}^2 = Q^2 (1 - x_{\bar\nu}) = m_\mu^2 (1 - x_{\bar\nu})$

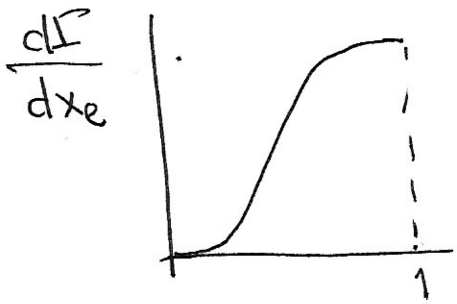
$$I = \frac{G_F^2}{16\pi^3} m_\mu^5 \int dx_e \int dx_\nu \quad (1-x_{\bar\nu})\, x_{\bar\nu}$$

d.) The integration region is $\displaystyle\int_0^1 dx_e \int_{1-x_e}^1 dx_{\bar\nu}$

then $$I \propto \int_{1-x_e}^1 dx_{\bar\nu}\, x_\nu (1-x_{\bar\nu}) = \int_0^{x_e} dz\, (1-z)\, z = \frac{x_e^2}{2} - \frac{x_e^3}{3}$$

then
$$\Gamma = \frac{G_F^2 m_\mu^5}{16\pi^3} \int_0^1 dx_e \left(\frac{x_e^2}{2} - \frac{x_e^3}{3}\right)$$

$$\frac{d\Gamma}{dx_e} = \frac{G_F^2 m_\mu^5}{32\pi^3} \left(x_e^2 - \frac{2}{3} x_e^3\right)$$



$x_e = \frac{2}{m_\mu} E_e$

[compare to the spectrum in Bardon et al. PRL 14, 449 (1965)]

e) Doing the integral

$$\int_0^1 dx_e \left(x_e^2 - \frac{2}{3} x_e^3\right) = \frac{1}{3} - \frac{2}{3}\frac{1}{4} = \frac{1}{6}$$

then
$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

to get a numerical value, use the value of G_F, the muon mass $m_\mu = 105.66$ MeV, and use $\hbar$ as a conversion factor from GeV to $\sec^{-1}$

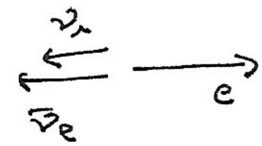
$$\Gamma = \left(\frac{1.166\times10^{-5}}{GeV^2}\right)^2 (0.10566\ GeV)^5 \frac{1}{192\pi^3}$$

$$\cdot \left[6.5821\times10^{-25}\ GeV\text{-}sec\right]^{-1}$$

$$= 4.57\times10^{5}/sec$$

$$\tau_\mu = \frac{1}{\Gamma_\mu} = 2.19\times10^{-6}\ sec.$$

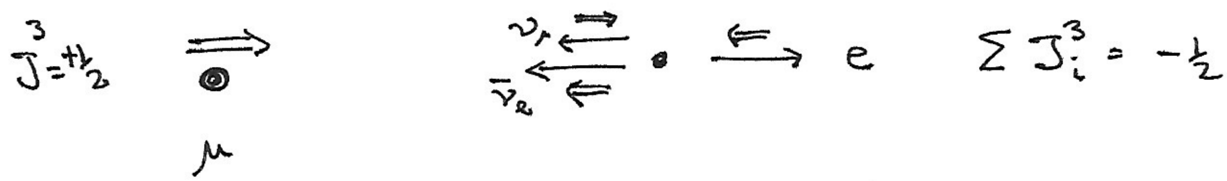
f.) If $x_e = 1$, the final state particles are in the configuration



If the electron is parallel to the muon spin direction, noticing from the ΔH that:

the massless e is Left-handed
the massless ν_μ is Left-handed
the massless $\bar\nu_e$ is Right handed

the angular momenta are



this is forbidden by angular momentum conservation.