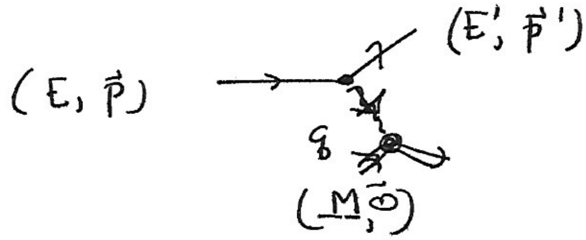
Physics 330 – Problem Set #5

Solutions

1.) The scattering process is



$\vec{q} = \vec{p} - \vec{p}\,'$ is the momentum transfer to the heavy particle. Then the final momentum of the heavy particle is $|\vec{q}| \ll M$ and its energy is

$$E_M' = M + \frac{|\vec{q}|^2}{2M} + \cdots = M\left(1 + \mathcal{O}\left(\frac{|\vec{q}|^2}{M^2}\right)\right)$$

Then 2-body phase space is

$$\int d\Pi_2 = \int \frac{d^3p'}{(2\pi)^3 2E'} \frac{d^3p_M'}{(2\pi)^3\, 2E_M'} (2\pi)^4 \delta(p + p_M - (p' + p_M'))$$

$$\approx \int \frac{d^3p'}{(2\pi)^3 2E'} \int \frac{d^3p_M'}{(2\pi)^3\, 2M} (2\pi)\, \delta(E_p - E_p')\, (2\pi)^3 \delta^{(3)}(\vec{p} - (\vec{p}\,' + \vec{q}))$$

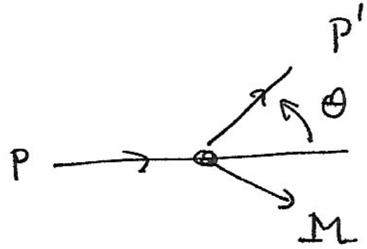
$$= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \frac{1}{2M} (2\pi)\, \delta(E_p - E_{p'})$$

$$\frac{dE_p'}{dp'} = \frac{p'}{E'} \qquad = \int \frac{d\cos\theta\, d\phi\; p^2}{(2\pi)^2 \cdot 4E'M} \frac{1}{\left|\left(\frac{dE_p'}{dp'}\right)\right|}\Bigg|_{p'=p}$$

$$= \int_{-1}^{1} \frac{d\cos\theta}{8\pi} \frac{p}{M} = \int \frac{d\cos\theta}{4\pi} \cdot \frac{p}{2M}$$

This answer is correct whether the electron moves relativistically or nonrelativistically.

b) A diagram of the momenta is



$$\vec{q} = (0,0,p) - (p\sin\theta, 0, p\cos\theta)$$

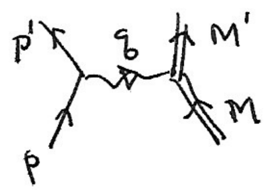
$$|\vec{q}|^2 = +(p\sin\theta)^2 + (p(1-\cos\theta))^2$$

$$= +p^2\sin^2\theta + p^2 - 2p^2\cos\theta + p^2\cos^2\theta$$

$$= +2p^2(1-\cos\theta) = +4p^2(\sin^2\theta/2)$$

$$q^2 = -|\vec{q}|^2 = -4p^2\sin^2\theta/2$$

c.) $i\mathcal{M} =$



$$= (-ie)^2 Z\, \bar{u}(p')\gamma^\mu u(p)\, \frac{-ig_{\mu\nu}}{q^2}\, \bar{u}(M')\gamma^\nu u(M)$$

In the nonrelativistic limit (m = electron mass)

$$\bar{u}^{s'}(p')\gamma^\mu u^{s}(p) = 2m\,(1,0,0,0)^\mu\, \delta^{s's}$$

$$\bar{u}^{s'}(M')\gamma^\nu u^{s}(M) = 2M\,(1,0,0,0)^\nu\, \delta^{s's}$$

so $$i\mathcal{M} = ie^2 Z\, 2m\, 2M\, \frac{1}{q^2}$$

The spins are preserved; we can ignore them.

The differential cross section is then

$$d\sigma = \frac{1}{2m\, 2M\, |v|} \int d\Pi_2 \; |\mathcal{M}|^2$$

$$= \frac{1}{2m\, 2M \cdot \frac{p}{m}} \int_{-1}^{1} \frac{d\cos\theta}{4\pi} \, \frac{p}{2M} \cdot e^4 Z^2 (2m \cdot 2M)^2 \times \left(\frac{1}{16 p^4 \sin^4\theta/2} \right)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{(4\pi\alpha)^2 Z^2}{(p/m)} \frac{2m}{} \frac{1}{4\pi} \frac{p}{16 p^4 \sin^4\theta/2}$$

$$= \frac{\pi\alpha^2 Z^2}{2} \frac{m^2}{p^4} \frac{1}{\sin^4\theta/2} \qquad \beta = v = \frac{p}{m}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Z^2}{2 m^2 \beta^4 \sin^4\theta/2}$$

d.) Now treat the electron (only) as relativistic.
For the heavy nucleus, we still have

$$\bar{u}^{(s')} \gamma^\nu u^{(s)} = 2M\, \delta^{ss'}$$

But for the electron, we will have a more complicated expression, involving the possibility of a spin flip. Take the initial electron to have momentum $\parallel \hat{3}$ and spin $\parallel \hat{3}$

$$p^\mu = (E, 0\, 0\, p)^\mu$$

$$u(p) = \begin{pmatrix} \sqrt{E-p}\, \binom{1}{0} \\ \sqrt{E+p}\, \binom{1}{0} \end{pmatrix}$$

the final electron will have momentum $p' = (E, p\sin\theta, 0\ p\cos\theta)$
of either helicity

$$u(p') = (+\text{ helicity}) = \begin{pmatrix} \sqrt{E-p}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\end{pmatrix} \\ \sqrt{E+p}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\end{pmatrix}\end{pmatrix}$$

or

$$(-\text{ helicity}) = \begin{pmatrix} \sqrt{E+p}\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix} \\ \sqrt{E-p}\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}\end{pmatrix}$$

the value of the Feynman diagram is

$$iM = $$

$$= (-ie)^2 Z\ \bar{u}(p')\gamma^\mu u(p)\ \frac{-i\eta_{\mu\nu}}{q^2} \cdot \bar{u}(M')\gamma^\nu u(M)$$

$$= ie^2 Z\ \bar{u}(p')\gamma^0 u(p)\ \frac{1}{q^2} \cdot 2M$$

$$= ie^2 Z\ u^\dagger(p')\, u(p)\ \frac{1}{q^2} \cdot 2M$$

so we only need to work out

$+$ helicity $\quad u^\dagger(p')u(p) = (E-p)\cos\theta/2 + (E+p)\cos\theta/2$

$$= 2E\cos\theta/2$$

$-$helicity $\quad u^\dagger(p')u(p) = [(E-p)(E+p)]^{1/2}(-\sin\theta/2) + [(E+p)(E-p)]^{1/2}(-\sin\theta/2)$

$$= -2m\sin\theta/2$$

square and sum over spins

$$4E^2\cos^2\theta/2 + 4m^2\sin^2\theta/2 = 4E^2 - 4p^2\sin^2\theta/2$$

assemble the pieces as above, using relativistic expressions for the electron

$$d\sigma = \frac{1}{2E\ \cancel{2M}} \frac{1}{(P/E)} \frac{d\cos\theta}{4\pi} \frac{P}{\cancel{2M}}$$

$$\cdot \frac{e^4 Z^2 \cancel{(2M)}^2 (4E^2 - 4p^2 \sin^2\theta/2)}{16 p^4 \sin^4\theta/2}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{(4\pi\alpha)^2 Z^2}{2\cancel{P} \cdot 4\pi} \cdot \cancel{P} \frac{1}{4p^4} (E^2 - p^2 \sin^2\theta/2) \frac{1}{\sin^4\theta/2}$$

using $\beta = P/E$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Z^2}{2} \frac{(1 - \beta^2 \sin^2\theta/2)}{E^2 \beta^4 (\sin^4\theta/2)}$$

2.) a.) In this case $\mathcal{H} = \mathcal{H}_0 + \int d^3x \, \frac{\lambda}{4}(\varphi^a)^2(\varphi^b)^2$

so

$$-i\int dt\, H_I = -i\int d^4x \, \frac{\lambda}{4}\, \varphi^a\varphi^a\varphi^b\varphi^b$$

summed over a and b $= 1 — N$

Imagine that this is contracted with 4 external φ's. There are $4!$ ways to contract, but these include 3 different ways to relate the indices. For example.

$$\langle \varphi^a(x)\,\varphi^b(y)\,\varphi^c(z)\,\varphi^d(w)\,\left(-i\int dt\, H_I\right)\rangle$$

includes

$$\langle \varphi^a(x)\,\varphi^b(y)\,\varphi^c(z)\,\varphi^d(w)\; -i\frac{\lambda}{4}\int d^4v\, \varphi^g(v)\varphi^g(v)\varphi^h(v)\varphi^h(v)\rangle$$

$$\int d^4v \left(-i\frac{\lambda}{4}\right) D_F(x-v)\,D_F(y-v)\,D_F(z-v)\,D_F(w-v)$$

$$\cdot\; \delta^{ag}\delta^{bg}\,\delta^{ch}\delta^{dh}$$

$$= \int d^4v\, D_F(x-v)\,D_F(y-v)\,D_F(z-v)\,D_F(w-v)$$

$$-i\frac{\lambda}{4}\,\delta^{ab}\delta^{cd}$$

The sum of possible contractions includes 3 such structures

$$\delta^{ab}\delta^{cd} \qquad \delta^{ac}\delta^{bd} \qquad \delta^{ad}\delta^{bc}$$

each appearing $4\cdot 2$ times

then the Feynman rule is

$$= -i\,\frac{\lambda}{4}\cdot(4\cdot 2)\left[\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right]$$

$$= -2i\lambda\left[\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right]$$

b.) the scattering matrix elements are then:

$\underline{\varphi^1\varphi^1 \to \varphi^1\varphi^1}$:

$$i\mathcal{M} = -6i\lambda$$

$\underline{\varphi^1\varphi^2 \to \varphi^1\varphi^2}$

$$i\mathcal{M} = -2i\lambda$$

$\underline{\varphi^1\varphi^1 \to \varphi^2\varphi^2}$

$$i\mathcal{M} = -2i\lambda$$

For the differential cross sections, we need the expression for 2-body phase space in the CM system

$$\int d\Pi_2 = \int \frac{d\cos\theta}{16\pi}\left(\frac{2p}{E_{cm}}\right) \qquad \underline{E_{cm} = 2E}$$

then

$$d\sigma = \frac{1}{2E\cdot 2E\cdot \underbrace{\left(2\frac{p}{E}\right)}_{\text{relative velocity } (v_1-v_2)}}\;\frac{d\cos\theta}{16\pi}\;\frac{p}{E}\;|\mathcal{M}|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{128\pi E^2}|\mathcal{M}|^2 = \frac{1}{32\pi E_{cm}^2}|\mathcal{M}|^2$$

then

$$\frac{d\sigma}{d\cos\theta} = \frac{9\lambda^2}{8\pi E_{cm}^2} \qquad \varphi^1\varphi^1 \to \varphi^1\varphi^1$$

$$= \frac{1}{8\pi E_{cm}^2}\lambda^2 \qquad \varphi^1\varphi^2 \to \varphi^1\varphi^2$$

$$= \frac{1}{8\pi E_{cm}^2}\lambda^2 \qquad \varphi^1\varphi^1 \to \varphi^2\varphi^2$$

c.) $\int_{-1}^{1} d\cos\theta = 2$ but for identical particles in the final state, we need to restrict the integral to in-equivalent final states

$$\sigma(\varphi^1\varphi^1 \to \varphi^1\varphi^1) = \frac{9}{8\pi}\frac{1}{E_{cm}^2}\lambda^2$$

$$\sigma(\varphi^1\varphi^2 \to \varphi^1\varphi^2) = \frac{2}{8\pi}\frac{1}{E_{cm}^2}\lambda^2$$

$$\sigma(\varphi^1\varphi^1 \to \varphi^2\varphi^2) = \frac{1}{8\pi}\frac{1}{E_{cm}^2}\lambda^2$$

d.) For the initial state $\varphi^1\varphi^1$ the final states can be $\varphi^1\varphi^1, \varphi^2\varphi^2 \dots \varphi^N\varphi^N$

$$\sigma_{tot} = \frac{9}{8\pi E_{cm}^2}\lambda^2 + \underbrace{\frac{1}{8\pi}\frac{1}{E_{cm}^2}\lambda^2 + \dots + \frac{1}{8\pi E_{cm}^2}\lambda^2}_{2 \,\dots\dots\, N}$$

so $$\sigma_{tot}(\varphi'\varphi') = \frac{(N+8)}{8\pi E_{cm}^2} \lambda$$

e.) Now consider the potential

$$V(\varphi) = -\frac{1}{2}\mu^2 (\varphi^a)^2 + \frac{\lambda}{4}[(\varphi^a)^2]^2$$

The minimum of the potential is found at

$$\frac{\partial V}{\partial \varphi^a} = -\mu^2 \varphi^a + \lambda \varphi^a (\varphi^b)^2$$

$$-\mu^2 + \lambda (\varphi^b)^2 = 0$$

or $$\varphi^b = \underset{\sim}{v}^b \left(\frac{\mu}{\sqrt{\lambda}}\right) = \underset{\sim}{v}^b_{\circ}\, v$$

where $\underset{\sim}{v}^b$ is an arbitrary unit vector

f.) Choose the solution $\varphi_0 = (0\ 0 \cdots 0, v)$
and expand $V(\varphi)$ about this point!

$$\varphi^a(x) = \pi^a(x) \qquad a < N$$

$$\varphi^N = v + \sigma(x)$$

$$V(\varphi) = -\frac{\mu^2}{2}\left[(\pi^a)^2 + (v+\sigma)^2\right]$$

$$+ \frac{\lambda}{4}\left[(\pi^a)^2 + (v+\sigma)^2\right]^2$$

$$= -\frac{\mu^2}{2}(\pi^a)^2 - \frac{\mu^2}{2}v^2 - \mu^2 v\sigma - \mu^2/_2\,\sigma^2$$

$$+ \frac{\lambda}{4}\left[v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4\right]$$

$$+ \frac{\lambda}{2}\left[(\pi^a)^2 v^2 + 2(\pi^a)^2 v\sigma + (\pi^a)^2\sigma^2\right]$$

$$+ \frac{\lambda}{4}\left[((\pi^a)^2)\right]^2$$

$$= -\frac{\mu^2}{2}v^2 + \frac{\lambda}{4}v^4 \qquad \text{(constant term in } V\text{)}$$

$$-\mu^2 v\sigma + \lambda v^3\sigma \qquad \text{(linear in } \sigma\text{)}$$

$$-\mu^2/_2\,\sigma^2 + \frac{3v^2\lambda}{2}\sigma^2 - \frac{\mu^2}{2}(\pi^a)^2 + \frac{\lambda}{2}v^2(\pi^a)^2$$

$$+ \lambda v\sigma^3 + \lambda v(\pi^a)^2\sigma$$

$$+ \frac{\lambda}{4}\left[\sigma^4 + 2(\pi^a)^2\sigma^2 + ((\pi^a)^2)^2\right]$$

Using $v^2\lambda = \mu^2$:

constant term $= -\frac{\mu^2}{4}v^2$ negative vacuum energy

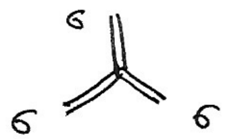
linear terms $= 0$ φ_0 is a minimum quadratic term: $\quad + \frac{2\mu^2}{2}\sigma^2 \quad + \quad 0$ for $(\pi^a)^2$

The fields π^a are massless, σ has the mass $m_\sigma^2 = 2\mu^2$

$$V(\varphi) = \frac{m_\sigma^2}{2}\sigma^2 + \lambda v \sigma^3 + \lambda v (\pi^a)^2 \sigma$$
$$+ \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{2}(\pi^a)^2\sigma^2 + \frac{\lambda}{4}\left[(\pi^a)^2\right]^2$$

g.) Put the σ mass term into H_0. Then we have

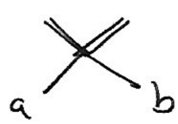
5 vertices:



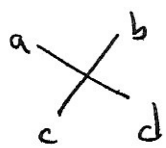
$= -6i\lambda v$

$= -2i\,\delta^{ab}\lambda v$

$= -6i\lambda$

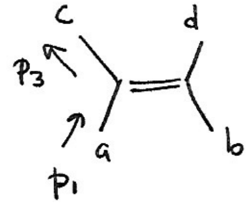


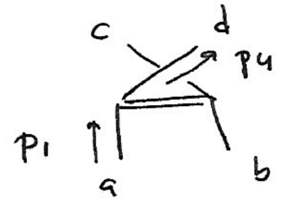
$= -2i\lambda$

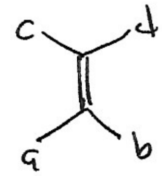


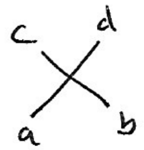
$= -2i\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)$

h) For $\pi^a \pi^b \to \pi^c \pi^d$ there are 4 leading-order Feynman diagrams:

$i\mathcal{M} =$  $(-2i\lambda v)^2 \delta^{ac}\delta^{bd} \dfrac{i}{(p_1-p_3)^2 - m_\sigma^2}$

$+$  $(-2i\lambda v)^2 \delta^{ad}\delta^{bc} \dfrac{i}{(p_1-p_4)^2 - m_\sigma^2}$

$+$  $(-2i\lambda v)^2 \delta^{ab}\delta^{cd} \dfrac{i}{(p_1+p_2)^2 - m_\sigma^2}$

$+$  $-2i\lambda\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)$

$$= (-2i\lambda)\Big[\delta^{ab}\delta^{cd}\left(1 + \frac{2\lambda v^2}{s - m_\sigma^2}\right) + \delta^{ac}\delta^{bd}\left(1 + \frac{2\lambda v^2}{t - m_\sigma^2}\right) + \delta^{ad}\delta^{bc}\left(1 + \frac{2\lambda v^2}{u - m_\sigma^2}\right)\Big]$$

but $2\lambda v^2 = 2\mu^2 = m_\sigma^2$

$$i\mathcal{M} = -2i\lambda\Big[\delta^{ab}\delta^{cd}\left(\frac{s}{s - m_\sigma^2}\right) + \delta^{ac}\delta^{bd}\left(\frac{t}{t - m_\sigma^2}\right) + \delta^{ad}\delta^{bc}\,\frac{u}{u - m_\sigma^2}\Big]$$

i.) $s + t + u = \frac{1}{2}\Big\{ (p_1+p_2)^2 + (p_3+p_4)^2$
$+ (p_1-p_3)^2 + (p_2-p_4)^2$
$+ (p_1-p_4)^2 + (p_2-p_3)^2 \Big\}$

$= \frac{1}{2}\Big\{ 3p_1^2 + 3p_2^2 + 3p_3^2 + 3p_4^2$
$+ 2p_1\cdot p_2 + 2p_3\cdot p_4 - 2p_1\cdot p_3 - 2p_2\cdot p_4$
$- 2p_1\cdot p_4 - 2p_2\cdot p_3 \Big\}$

$= \frac{1}{2}\Big\{ 2p_1^2 + 2p_2^2 + 2p_3^2 + 2p_4^2$
$+ (p_1+p_2-p_3-p_4)^2 \Big\}$

but $p_1 + p_2 - p_3 - p_4 = 0$ momentum conservation

$p_1^2 = p_2^2 = p_3^2 = p_4^2$ since the π^a are massless

$= 0$

j.) Expand the result on p. 12 for $m_\sigma^2 \gg s, t, u$

$$i\mathcal{M} = +\frac{2i\lambda}{m_\sigma^2}\Big[\delta^{ab}\delta^{cd}\Big(s + \frac{s^2}{m_\sigma^2} + \ldots\Big) + \delta^{ac}\delta^{bd}\Big(t + \frac{t^2}{m_\sigma^2} + \ldots\Big)$$
$$+ \delta^{ad}\delta^{bc}\Big(u + \frac{u^2}{m_\sigma^2} + \ldots\Big)\Big]$$

This expression vanishes at $s = t = u = 0$

For $N = 2$, $a, b, c, d = 1$ only so

$$i\mathcal{M} = + \frac{2i\lambda}{m_\sigma^2} \left[s + t + u + \mathcal{O}\left((s,t,u)^2\right) \right]$$

$$= 0 + \mathcal{O}\left((s,t,u)^2\right)$$