Physics 330 — Problem Set #4

Solutions

1.) a.) The amplitude for the vacuum $|0\rangle$ in the far past to evolve to the vacuum $|0\rangle$ in the far future is

$$a_0 = \frac{\langle \Omega | T\{\exp[-ig\int dt\, H_I(t)]\} |\Omega\rangle}{\langle \Omega | \Omega\rangle}$$

Here $|\Omega\rangle = |0\rangle$ and $H_I(t) = -\int d^3x\, j(t,\vec{x})\, \phi_I(t,\vec{x})$. Then the probability is

$$P(0) = |a|^2 = \left| \langle 0 | T\left\{ \exp\left[+ig\int dt \int d^3x\, j(x)\, \phi_I(x)\right]\right\} |0\rangle \right|^2$$

The probability for a particle of momentum p to be produced is

$$|a(p)|^2 = \left| \langle p | T\left\{ \exp\left[ig\int d^4x\, j(x)\, \phi_I(x)\right]\right\} |0\rangle \right|^2$$

b.) In order g

$$\begin{aligned} a(p) &= \langle p|\, ig\int d^4x\, j(x)\, \phi(x)\, |0\rangle \\ &= \langle p|\, ig\int d^4x \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\, a_q^\dagger\, e^{+iq\cdot x}\, j(x)\, |0\rangle \\ &= \langle p|\, ig\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\, a_q^\dagger\, (\tilde{j}(+q))\, |0\rangle \end{aligned}$$

In relativistic normalization $\langle p| = \langle 0| a_p \sqrt{2E_p}$

so $\quad \mathcal{A}(p) = \langle 0| a_p \sqrt{2E_p} \cdot ig \cdot \int \frac{d^3 q}{(2\pi)^3}\, a_q^\dagger\, \tilde{j}(q)\, |0\rangle$

$= ig\, \tilde{j}(p)$

Another way: $\quad \langle p|\phi(x)|0\rangle = e^{+ip\cdot x}$

$$\mathcal{A}(p) = ig \int d^4x\, \langle p|\phi(x)|0\rangle\, j(x)$$
$$= ig \int d^4x\, e^{+ip\cdot x}\, j(x) = ig\, \tilde{j}(p)$$

Summing over momenta:

$$P(1) = g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \quad \text{to this order in } g$$
$$= N$$

c.) Evaluating $P(0) = |\mathcal{A}_0|^2$ in perturbation theory

$$\mathcal{A}_0 = \langle 0| 1 + (ig) \int d^4x\, \phi(x) j(x)$$
$$+ \frac{1}{2}(ig)^2 \int d^4x\, d^4y\, T[\phi(x)\phi(y)]\, j(x) j(y) + \cdots |0\rangle$$

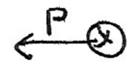
$$= 1 + 0 + \frac{1}{2}(-g^2) \int d^4x\, d^4y\, \overbrace{\phi(x)\phi(y)}\, j(x) j(y) + \cdots$$

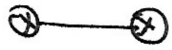
now $\quad \overbrace{\phi(x)\phi(y)} = \begin{cases} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} & x^0 > y^0 \\ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip(y-x)} & y^0 > x^0 \end{cases}$

For $y^0 > x^0$, interchange the dummy variables x and y.

$$= 1 + \left(\frac{-g^2}{2}\right) \int d^4x\, d^4y \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x} j(x)\, e^{+ip\cdot y} j(y) + \cdots$$

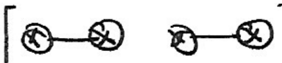
$$= 1 - \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 + \cdots$$

We can recognize the Feynman rule  $= ig\,\tilde{j}(p)$

but also  $= \frac{1}{2}(ig)^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2$

($\frac{1}{2}$ is a symmetry factor) $\quad = -\frac{1}{2} N$

d.) The complete diagrammatic expansion for $P(0)$ is

$a_0 = 1 +$ [] + []

+ [] + ...

The diagram with n 's has the symmetry factor

$$\underbrace{\left(\tfrac{1}{2}\right)^n}_{\text{already included above}} \times \frac{1}{n!}$$

then

$$P(0) = \left| \left[1 + \left(-\tfrac{1}{2}N\right) + \tfrac{1}{2!}\left(-\tfrac{1}{2}N\right)^2 + \tfrac{1}{3!}\left(-\tfrac{1}{2}N\right)^3 + \cdots \right] \right|^2$$

$$= \left| \exp\left[-\tfrac{1}{2}N\right] \right|^2 = e^{-N}$$

e.) To all orders in g, the amplitude for producing 1 particle is

$$\mathcal{A}(p) =$$ 

$$+ \cdots$$

$$= i g \tilde{j}(p) \left[1 - \tfrac{1}{2}N + \tfrac{1}{2!}\left(\tfrac{1}{2}N\right)^2 + \cdots \right]$$

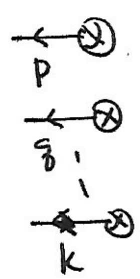
$$= i g \tilde{j}(p)\, e^{-\frac{1}{2}N}$$

so the probability to produce 1 particle is

$$P(1) = \left(g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \right) e^{-N}$$

$$= N e^{-N}$$

f.) The first Feynman diagram for producing n particles is



$$= (ig\,\tilde{j}(p))(ig\,\tilde{j}(q)) \cdots (ig\,\tilde{j}(k))$$

square this and ~~integrate~~ over states:

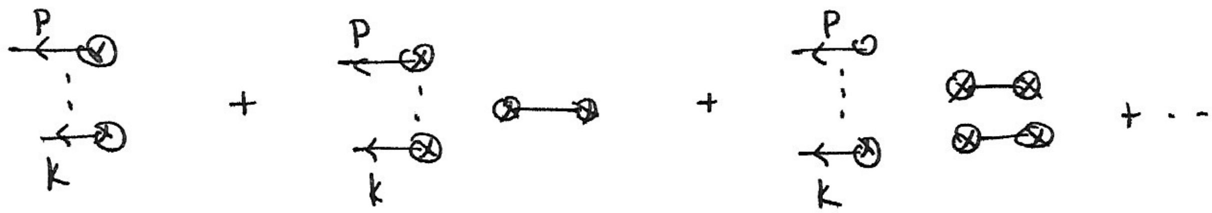
$$P(n) = \int \frac{d^3p\, d^3q \cdots d^3k}{(2\pi)^3 2E_p (2\pi)^3 2E_q \cdots (2\pi)^3 2E_k}\; g^2 |\tilde{j}(p)|^2\, g^2 |\tilde{j}(q)|^2 \cdots g^2 |\tilde{j}(k)|^2$$

but Klein - Gordon particles are indistinguishable. The states (p, q) and (q, p), $(p\,q\,k)\ (k\,q\,p)\ (p\,k\,q)$ etc. are identical and should not be counted multiple times. So, to leading order

$$P(n) = \frac{1}{n!}\left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{j}(p)|^2\right] \cdots \left[\int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}|\tilde{j}(k)|^2\right.$$

$$= \frac{N^n}{n!}$$

The full set of Feynman diagrams is



giving

$$P(n) = \frac{N^n}{n!}\left|\left[1 - \frac{1}{2}N + \frac{1}{2!}\left(\frac{1}{2}N\right)^2 + \cdots\right]\right|^2$$

$$= \frac{N^n}{n!}e^{-N}$$

g.) $$\sum_{n=0}^{\infty} P(n) = \left(1 + N + \frac{N^2}{2} + \cdots\right)e^{-N}$$

$$= e^{N}e^{-N} = 1$$

$$\sum_{n=0}^{\infty} n P(n) = \sum_{n=0}^{\infty} n \frac{N^n}{n!}e^{-N} = \frac{\partial}{\partial\lambda}\sum \frac{(\lambda N)^n}{n!}e^{-N}\Big|_{\lambda=1}$$

$$= \frac{\partial}{\partial\lambda}e^{\lambda N}\Big|_{\lambda=1}e^{-N} = N$$

h) $$\tilde{j}(p) = \int d^4x \, e^{+ip\cdot x} \, y(x)$$

$$= \int_{-\infty}^{0} d\tau \, e^{ip\cdot k\tau} + \int_{0}^{\infty} d\tau \, e^{ip\cdot k'\tau}$$

regulate appropriately

$$= \int_{-\infty}^{0} d\tau \, e^{ip\cdot k\tau} e^{+\epsilon\tau} + \int_{0}^{\infty} d\tau \, e^{ip\cdot k'\tau} e^{-\epsilon\tau}$$

$$= \int_{0}^{\infty} d\tau \, e^{-ip\cdot k\tau - \epsilon\tau} + \int_{0}^{\infty} d\tau \, e^{+ip\cdot k'\tau} e^{-\epsilon\tau}$$

$$= \frac{1}{ip\cdot k + \epsilon} + \frac{1}{-ip\cdot k' + \epsilon}$$

$$|\tilde{j}(p)|^2 = \left| \frac{1}{p\cdot k'} - \frac{1}{p\cdot k} \right|^2$$

$$N = g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left| \frac{1}{p\cdot k'} - \frac{1}{pk} \right|^2$$

i) Let $k = (E_k, \vec{k})$ $\quad k' = (E_k', \vec{k}')$ $\quad p = (p, \vec{p})$

$$N = g^2 \int \frac{dp \, p^2 \, d\Omega}{(2\pi)^3 \, 2p} \left| \frac{1}{[p(E_k' - k'\cos\theta')]} - \frac{1}{p(E_k - k\cos\theta)} \right|^2$$

where $\cos\theta = \hat{p}\cdot\hat{k}$ and $\cos\theta' = \hat{p}\cdot\hat{k}'$

we can factor out p:

$$N = \frac{g^2}{16\pi^3}\int \frac{dp}{p}\, d\Omega_p \left| \frac{1}{E'_k - k'\cos\theta'} - \frac{1}{E_k - k\cos\theta} \right|^2$$

the integral over p normally runs from 0 to ∞. In a real situation, the incoming particle with energy E_K cannot radiate more energy than it has, so the dp integral will have an upper bound. However, the lower bound really is 0, so this integral

$$\int_0^{p_{max}} \frac{dp}{p}$$

really is infinite. An infinite number of particles are radiated in this case. This applies, for example, to emission of photons by an accelerated charge.

However, most of these particles are at low energy. The energy radiated is

$$\mathcal{E} = \frac{g^2}{16\pi^3}\int_0^{p_{max}} \underbrace{\frac{dp}{p}\cdot p}_{\text{finite w. lower limit = 0 !}} \cdot\, d\Omega_p \left| \frac{1}{E'_k - k'\cos\theta'} - \frac{1}{E_k - k\cos\theta} \right|^2$$

Notice that the integrand has peaks when $\cos\theta' = 1$ or $\cos\theta = 1$. So most of the particles are radiated along the incoming and outgoing directions of the radiator.