Physics 330 — Problem Set #3

Solutions

a.) Write the positronium wavefunction as

$$|B\rangle = \int \frac{d^3p}{(2\pi)^3}\, \psi(\vec{p})\, S_{s_1 s_2}\, a^{\dagger s_1}_{p}\, b^{\dagger s_2}_{-p}\, |0\rangle$$

$\psi(p)$ is the Fourier transform of a spatial wavefunction

$\psi(x) \sim f(r)\, Y_{\ell m}(\Omega)$. Then

$$\psi(-\vec{p}) = (-1)^{\ell}\, \psi(\vec{p})$$

$S_{s_1 s_2}$ is the spin wavefunction. For $S=0$ this is

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \quad \text{i.e.} \quad S_{s_1 s_2} = \frac{1}{\sqrt{2}}\, \epsilon_{s_1 s_2}$$

For $S=1$ this is

$$|\uparrow\uparrow\rangle \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad |\downarrow\downarrow\rangle$$

so

$$S_{s_1 s_2} = -\,S_{s_2 s_1} \quad \text{for } S=0$$
$$S_{s_1 s_2} = +\,S_{s_2 s_1} \quad \text{for } S=1$$

Acting P on $|B\rangle$ gives, using $P a^s_{\vec{p}} P = a^s_{-\vec{p}}$ $\quad P b^s_{\vec{p}} P = -b^s_{-\vec{p}}$

$$(-1)\int \frac{d^3p}{(2\pi)^3}\, \psi(\vec{p})\, S_{s_1 s_2}\, a^{\dagger s_1}_{-\vec{p}}\, b^{\dagger s_2}_{+\vec{p}}\, |0\rangle = (-1)\int \frac{d^3p}{(2\pi)^3}\, \psi(-\vec{p})\, S_{s_1 s_2}\, a^{\dagger s_1}_{p}\, b^{\dagger s_2}_{-p}\, |0\rangle$$

$$= (-1)(-1)^{\ell}\, |B\rangle$$

Acting C on $|B\rangle$ spins, using $C a_p^s C = b_p^s \quad C b_p^s C = a_p^s$,

$$\int \frac{d^3p}{(2\pi)^3} \psi(\vec{p}) S_{s_1 s_2}\, b_p^{\dagger s_1} a_{-p}^{\dagger s_2} |0\rangle = - \int \frac{d^3p}{(2\pi)^3} \psi(\vec{p}) S_{s_1 s_2}\, a_{-p}^{\dagger s_2} b_p^{\dagger s_1} |0\rangle$$

$$= - \int \frac{d^3p}{(2\pi)^3} \psi(-\vec{p}) S_{s_2 s_1}\, a_p^{s_1 \dagger} b_p^{s_2 \dagger} |0\rangle$$

$$= (-1)\cdot(-1)^{l} \cdot \begin{cases} (-1) & S=0 \\ +1 & S=1 \end{cases} = (-1)^{l+S}$$

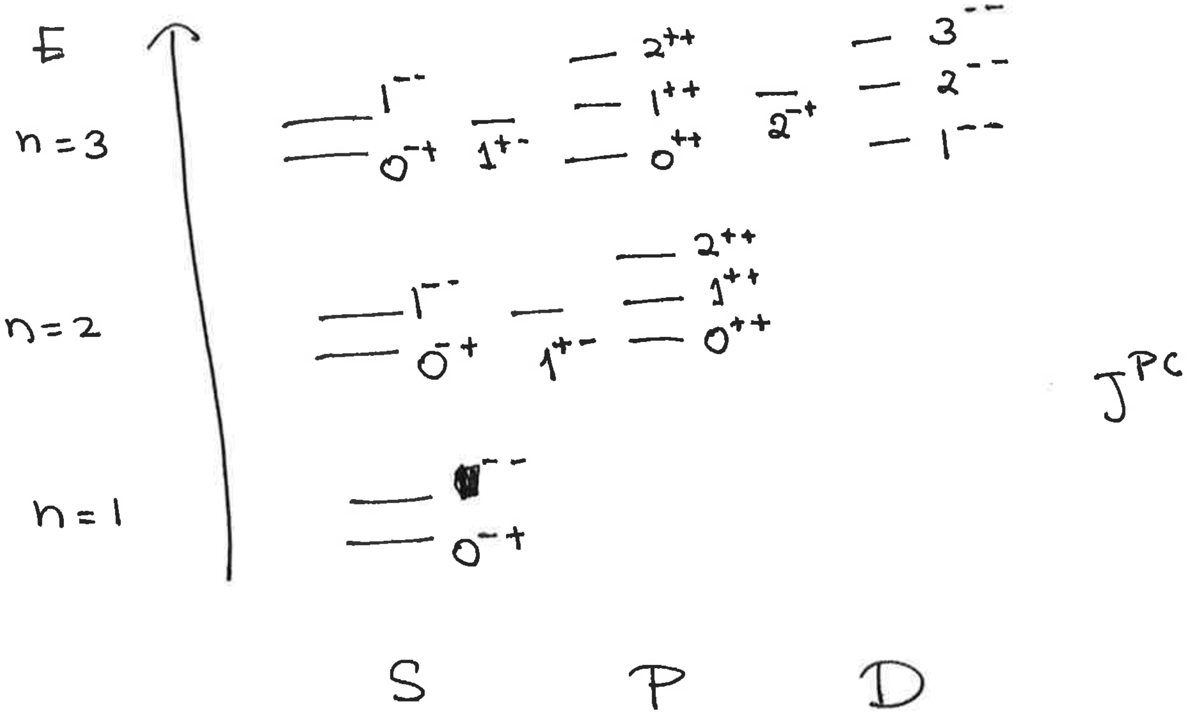
so

$$P = (-1)^{L+1} \qquad C = (-1)^{L+S}$$

b.) For the various positronium levels, J is

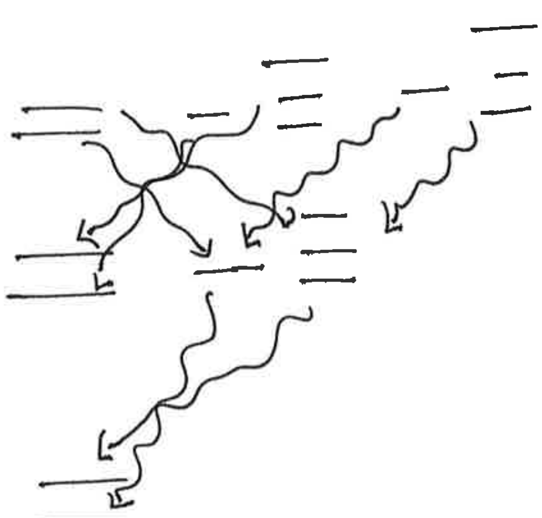
$n=1$	$1S$ $L=0$	$S=0$	$J=0$
		$S=1$	$J=1$
$n=2$	$2S$ $L=0$	$S=0$	$J=0$
		$S=1$	$J=1$
	$2P$ $L=1$	$S=0$	$J=1$
		$S=1$	$J=0,1,2$
$n=3$	$3S$ $L=0$	$S=0$	$J=0$
		$S=1$	$J=1$
	$3P$ $L=1$	$S=0$	$J=1$
		$S=1$	$J=0,1,2$
	$3D$ $L=2$	$S=0$	$J=2$
		$S=1$	$J=1,2,3$

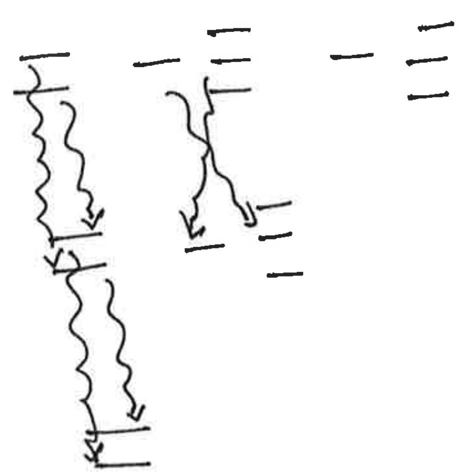
graphically



c.) Electric dipole (E1) transitions

Magnetic dipole (M1) transitions





(I show just enough of these for you to see the pattern. Transitions from states to states with the same n depend on whether they're allowed by the fine structure splittings.)

d.) the 1S $S=0$ state has $C=+1$

the 1S $S=1$ state has $C=-1$

so

the 1S $S=0$ (para-positronium) annihilates to 2 photons

the 1S $S=0$ (ortho-positronium) annihilates to 3 photons

2.) Before beginning, note that

$\bar\psi\psi \qquad \bar\psi\gamma^\mu\psi \qquad \bar\psi\gamma^\mu\gamma^5\psi$ are Hermitian

$$[\bar\psi\gamma^\mu\gamma^5\psi]^\dagger = \psi^\dagger\gamma^5(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^0(-\gamma^5)\gamma^\mu\psi = \bar\psi\gamma^\mu\gamma^5\psi$$

but $\bar\psi[\gamma^\mu,\gamma^\nu]\psi \qquad \bar\psi\gamma^5\psi$ are anti-Hermitian

$\therefore$

$$[\bar\psi[\gamma^\mu,\gamma^\nu]\psi]^\dagger = \bar\psi[\gamma^\nu,\gamma^\mu]\psi = -\bar\psi[\gamma^\mu,\gamma^\nu]\psi$$

$$[\bar\psi\gamma^5\psi]^\dagger = \psi^\dagger\gamma^5\gamma^0\psi = -\bar\psi\gamma^5\psi$$

so we include

$$i\bar\psi\gamma^5\psi \qquad \frac{i}{2}\bar\psi[\gamma^\mu,\gamma^\nu]\psi = \bar\psi\sigma^{\mu\nu}\psi$$

in our list of bilinears.

Under P: $\quad \psi \to \gamma^0\psi \qquad \bar\psi \to \bar\psi\gamma^0$, then

$$\bar\psi\psi \to \bar\psi\gamma^0\gamma^0\psi = \bar\psi\psi$$

$$i\bar\psi\gamma^5\psi \to i\bar\psi\gamma^0\gamma^5\gamma^0\psi = -i\bar\psi\gamma^5\psi$$

$$\bar\psi\gamma^\mu\psi \to \bar\psi\gamma^0\gamma^\mu\gamma^0\psi = \left\{\begin{matrix}\bar\psi\gamma^0\psi \\ -\bar\psi\gamma^i\psi\end{matrix}\right\} = (-1)^\mu\,\bar\psi\gamma^\mu\psi$$

$$\bar\psi\gamma^\mu\gamma^5\psi \to \bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi = \bar\psi(-1)^\mu\gamma^\mu(-\gamma^5)\psi = -(-1)^\mu\bar\psi\gamma^\mu\gamma^5\psi$$

$$\bar\psi\sigma^{\mu\nu}\psi \to \bar\psi\gamma^0\frac{i}{2}[\gamma^\mu,\gamma^\nu]\gamma^0\psi = (-1)^\mu(-1)^\nu\,\bar\psi\frac{i}{2}[\gamma^\mu,\gamma^\nu]\psi$$

$$= (-1)^\mu(-1)^\nu\,\bar\psi\sigma^{\mu\nu}\psi$$

$$\partial^\mu \to \left\{\begin{matrix}\partial/\partial t^0 \\ -\partial/\partial x^i\end{matrix}\right\} \to (-1)^\mu\,\partial^\mu$$

Under T : $\quad \psi \to (\gamma^1\gamma^3)\psi \qquad \bar\psi \to \bar\psi(-\gamma^1\gamma^3)$

and T is anti-unitary $\quad c \to c^*$

$\bar\psi\psi \to \bar\psi(-\gamma^1\gamma^3)(\gamma^1\gamma^3)\psi = \bar\psi(+\gamma^1\gamma^1\gamma^3\gamma^3)\psi = +\bar\psi\psi$

$i\bar\psi\gamma^5\psi \to (-i)\bar\psi(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3) = -i\bar\psi(-\gamma^1\gamma^3)(\gamma^1\gamma^3)\gamma^5$

$= -i\bar\psi\gamma^5\psi$

$\bar\psi\gamma^\mu\psi \to \bar\psi(-\gamma^1\gamma^3)(\gamma^\mu)^*\gamma^1\gamma^3\psi = \begin{cases}\bar\psi(-\gamma^1\gamma^3)(\gamma^1\gamma^3)\gamma^0\psi \\ \bar\psi(-\gamma^1\gamma^3)(\gamma^1\gamma^3)(-\gamma^{1,3})\psi \\ \bar\psi(-\gamma^1\gamma^3)(\gamma^1\gamma^3)(-\gamma^2)\psi\end{cases}$

$= (-1)^\mu\bar\psi\gamma^\mu\psi$

$\bar\psi\gamma^\mu\gamma^5\psi \to \bar\psi(-\gamma^1\gamma^3)(\gamma^\mu)^*(\gamma^5)^*(\gamma^1\gamma^3)\psi = \bar\psi(-\gamma^1\gamma^3)(\gamma^\mu)^*(\gamma^1\gamma^3)\gamma^5\psi$

$= (-1)^\mu\bar\psi\gamma^\mu\gamma^5\psi$

$\bar\psi\sigma^{\mu\nu}\psi \to \bar\psi(-\gamma^1\gamma^3)(\frac{-i}{2})[(\gamma^\mu)^*,(\gamma^\nu)^*](\gamma^1\gamma^3)\psi$

$= (-\frac{i}{2})(-1)^\mu(-1)^\nu\bar\psi[\gamma^\mu,\gamma^\nu]\psi = -(-1)^\mu(-1)^\nu\bar\psi\sigma^{\mu\nu}\psi$

$\partial^\mu \to \begin{Bmatrix}-\partial/\partial x^0 \\ +\partial/\partial x^i\end{Bmatrix} = -(-1)^\mu\partial^\mu$

Under C : $\quad \psi \to -i(\bar\psi\gamma^0\gamma^2)^T \qquad \bar\psi \to -i(\gamma^0\gamma^2\psi)^T$

$\bar\psi\psi \to (-i)^2(\gamma^0\gamma^2\psi)^T(\bar\psi\gamma^0\gamma^2)^T$

$= (-i)^2(-1)\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi = -\bar\psi(\gamma^0)^2(\gamma^2)^2\psi = +\bar\psi\psi$

$i\bar\psi\gamma^5\psi \to (-i)^2(\gamma^0\gamma^2\psi)^T_a\gamma^5_{ab}(\bar\psi\gamma^0\gamma^2)^T_b$

$= (-i)^2(-1)(\bar\psi\gamma^0\gamma^2)(\gamma^5)^T(\gamma^0\gamma^2\psi)$

$= +\bar\psi\gamma^5(\gamma^0\gamma^2)(\gamma^0\gamma^2)\psi = +\bar\psi\gamma^5\psi$

$$\bar\Psi\gamma^\mu\Psi \to (-i)^2(\gamma^0\gamma^2\Psi)^T\gamma^\mu(\bar\Psi\gamma^0\gamma^2)^T$$

$$= (-i)^2(-1)\ \bar\Psi\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\Psi$$

$$= \begin{cases} \bar\Psi\gamma^0\gamma^2\gamma^0\gamma^0\gamma^2\Psi = -\bar\Psi\gamma^0\Psi \\ \bar\Psi\gamma^0\gamma^2(-\gamma^{1,3})\gamma^0\gamma^2\Psi = -\bar\Psi\gamma^{1,3}(\gamma^0\gamma^2\gamma^0\gamma^2)\Psi = -\bar\Psi\gamma^{1,3}\Psi \\ \bar\Psi\gamma^0\gamma^2(+\gamma^2)\gamma^0\gamma^2\Psi = +\bar\Psi\gamma^0\gamma^2\gamma^2\gamma^0\gamma^2\Psi = -\bar\Psi\gamma^2\Psi \end{cases}$$

$$= -\bar\Psi\gamma^\mu\Psi$$

$$\bar\Psi\gamma^\mu\gamma^5\Psi \to (-i)^2(\gamma^0\gamma^2\Psi)^T_a(\gamma^\mu\gamma^5)_{ab}(\bar\Psi\gamma^0\gamma^2)_b$$

$$= (-i)^2(-1)\ \bar\Psi\gamma^0\gamma^2(\gamma^5)^T(\gamma^\mu)^T\gamma^0\gamma^2\Psi$$

$$= (+1)\ \bar\Psi\gamma^5\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\Psi$$

$$= -\bar\Psi\gamma^5\gamma^\mu\Psi = +\bar\Psi\gamma^\mu\gamma^5\Psi$$

$$\partial_\mu \to \partial_\mu \text{ or } +\partial_\mu$$

This reproduces the first three lines of the table.

Multiply down we find

$$\bar\Psi\Psi,\quad i\bar\Psi\gamma^5\Psi,\quad \bar\Psi\sigma^{\mu\nu}\Psi \to +$$

$$\bar\Psi\gamma^\mu\Psi,\quad \bar\Psi\gamma^\mu\gamma^5\Psi,\quad \partial_\mu \to 1$$

in a product with all Lorentz indices contracted

$$\to +.$$

3.) a.)

$$H = \int d^3x \, (\psi_L^\dagger, \psi_R^\dagger)\, \gamma^0 \gamma^j \nabla^j \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$= \int d^3x \, (\psi_L^\dagger \; \psi_R^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-i) \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \nabla^j \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$= \int d^3x \left\{ \psi_L^\dagger (i\sigma^j \nabla^j) \psi_L + \psi_R^\dagger (-i\sigma^j \nabla^j) \psi_R \right\}$$

so

$$H_L = \int d^3x \; \psi_L^\dagger \, i\vec{\sigma}\cdot\vec{\nabla} \, \psi_L \qquad H_R = \int d^3x \; \psi_R^\dagger (-i\vec{\sigma}\cdot\vec{\nabla}) \psi_R$$

$$H = H_L + H_R$$

b.)

$$i\frac{\partial}{\partial t}\psi_L(x) = [\psi_L(x), H_L] = \left[\psi_L(x), \int d^3y \; \psi_L^\dagger \, i\vec{\sigma}\cdot\vec{\nabla} \, \psi_L(y)\right]$$

$$= i\vec{\sigma}\cdot\vec{\nabla}\psi_L$$

so

$$i\frac{\partial}{\partial x^0}\psi_L - i\vec{\sigma}\cdot\vec{\nabla}\psi_L = 0$$

$$(i\bar{\sigma}^\mu \partial_\mu)\psi_L = 0 \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})^\mu$$

similarly

$$i\frac{\partial}{\partial t}\psi_R = [\psi_R, H_R] = -i\vec{\sigma}\cdot\vec{\nabla}\psi_R$$

so

$$i\frac{\partial}{\partial x^0}\psi_R + i\vec{\sigma}\cdot\vec{\nabla}\psi_R = 0$$

or

$$i\sigma^\mu \partial_\mu \psi_R = 0 \qquad \sigma^\mu = (1, \vec{\sigma})^\mu$$

c.) An infinitesimal Lorentz transformation on ψ is

$$\psi \to \left[1 - i\tfrac{1}{2}\omega_{\mu\nu} S^{\mu\nu}\right]\psi$$

with $\quad \omega_{ij} = \epsilon_{ijk}\theta^k \qquad \omega_{0i} = \eta_i$

$$\psi \to \left[1 - i\tfrac{1}{2}\theta^k \epsilon_{ijk} S^{ij} - i\eta_i S^{0i}\right]\psi$$

$$= \left[1 - i\begin{pmatrix} \vec{\theta}\cdot\vec{\sigma}/2 & \\ & \vec{\theta}\cdot\vec{\sigma}/2 \end{pmatrix} + \begin{pmatrix} -\vec{\eta}\cdot\vec{\sigma}/2 & \\ & \vec{\eta}\cdot\vec{\sigma}/2 \end{pmatrix}\right]\psi$$

or

$$\psi_L \to (1 - i\vec{\theta}\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2)\psi_L$$

$$\psi_R \to (1 - i\vec{\theta}\vec{\sigma}/2 + \vec{\eta}\vec{\sigma}/2)\psi_R$$

the 2nd equation implies

$$-i\sigma^2\psi_R^* \to (-i\sigma^2)(1 + i\vec{\theta}\vec{\sigma}^*/2 + \vec{\eta}\cdot\vec{\sigma}^*/2)\psi_R^*$$

$$(-i\sigma^2)(\vec{\sigma})^* = (-\vec{\sigma})(-i\sigma^2)$$

$$-i\sigma^2\psi_R^* \to (1 - i\vec{\theta}\vec{\sigma}/2 - \vec{\eta}\cdot\vec{\sigma}/2)(-i\sigma^2)\psi_R^*$$

$$\psi_L' = (-i\sigma^2)\psi_R^*$$

$$(+i\sigma^2)\psi_L' = \psi_R^*$$

$$(i\sigma^2)\psi_L'^* = \psi_R$$

then
$$\psi_R = [\psi_L'^{\dagger}(-i\sigma^2)]^T$$
$$\psi_R^{\dagger} = [(i\sigma^2)\psi_L']^T$$

$$\int d^3x\, \psi_R^{\dagger}(-i\vec{\sigma}\cdot\vec{\nabla})\psi_R$$
$$= \int d^3x\, (i\sigma^2\psi_L')^T(-i\vec{\sigma}\cdot\vec{\nabla})[\psi_L'^{\dagger}(-i\sigma^2)]^T$$
$$= \int d^3x\, (-\vec{\nabla}\cdot i\sigma^2\psi_L')^T(-i\vec{\sigma})[\psi_L'^{\dagger}(-i\sigma^2)]^T$$
$$= \int d^3x\, (-1)^2\psi_L'^{\dagger}(-i\sigma^2)(-i\vec{\sigma})^T(i\sigma^2\vec{\nabla}\psi_L')$$

since $\vec{\sigma}^T = \sigma^*$, $(-i\sigma^2)\vec{\sigma}^T = -\vec{\sigma}(-i\sigma^2)$

$$= \int d^3x\, (-1)^3\,\psi_L'^{\dagger}(i\vec{\sigma})\underbrace{(-i\sigma^2)(i\sigma^2)}_{1}\vec{\nabla}\psi_L'$$
$$= \int d^3x\, \psi_L'^{\dagger}(+i\vec{\sigma}\cdot\vec{\nabla})\psi_L' \quad \text{a copy of } H_L\,!$$

e.)
$$\bar{\psi}\psi = (\psi_L^{\dagger}, \psi_R^{\dagger})\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\psi_L\\\psi_R\end{pmatrix}$$
$$= \psi_L^{\dagger}\psi_R + \psi_R^{\dagger}\psi_L$$
$$= \psi_L^{\dagger}(+i\sigma^2)\psi_L'^{*} + \psi_L'^{T}(-i\sigma^2)\psi_L$$

$$(i\sigma^2)_{ab} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = \epsilon_{ab}$$

so
$$= \psi_{La}^{*}\epsilon_{ab}\psi_{Lb}'^{*} - \psi_{La}'\,\epsilon_{ab}\psi_{Lb}$$

Notice that

$$\psi'_{La}\,\varepsilon_{ab}\,\psi_{Lb} = -\psi_{Lb}\,\varepsilon_{ab}\,\psi'_a$$
$$= +\psi_{La}\,\varepsilon_{ab}\,\psi'_{Lb}$$

and similarly for ψ_L^*, $\psi_L'^*$

so

$$\Delta H = \int d^3x\; m\bar{\Psi}\Psi = \int d^3x\; m\left[-\psi_{La}\,\varepsilon_{ab}\,\psi'_{Lb} + \psi^*_{La}\,\epsilon_{ab}\,\psi'^*_{Lb}\right]$$

Under a Lorentz transformation

$$\psi_{La}\,\varepsilon_{ab}\,\psi'_{Lb} \to \left((1-i\vec{\theta}\cdot\vec{\sigma}/2-\vec{\eta}\cdot\vec{\sigma}/2)\,\psi_L\right)_a (i\sigma^2)_{ab}\left((1-i\vec{\theta}\cdot\vec{\sigma}/2-\vec{\eta}\cdot\vec{\sigma}/2)\,\psi'_L\right)_b$$

$$= \psi_L^T\,(1-i\vec{\theta}\cdot\vec{\sigma}^T/2-\eta\cdot\vec{\sigma}^T/2)\,(i\sigma^2)\cdot(1-i\vec{\theta}\cdot\vec{\sigma}/2-\vec{\eta}\cdot\vec{\sigma}/2)\,\psi'_L$$

$$= \psi_L^T\,(1-i\vec{\theta}\cdot\vec{\sigma}/2-\vec{\eta}\cdot\vec{\sigma}/2)\cdot(1+i\vec{\theta}\cdot\vec{\sigma}^*/2+\vec{\eta}\cdot\vec{\sigma}^*/2)\,(i\sigma^2)\,\psi'_L$$

since $\vec{\sigma}^* = \vec{\sigma}^T$, the two factors are inverses of one another!

$$= \psi_L^T\,(i\sigma)\,\psi'_L = \psi_{La}\,\varepsilon_{ab}\,\psi'_{Lb}$$

and the same argument works for $\psi_L^*\,\psi_L'^*$.

f.) The Hamiltonian of the Majorana theory is

$$\mathcal{H} = \int d^3x \left\{ \psi_L^\dagger (i\vec{\sigma}\cdot\vec{\nabla})\psi_L - \tfrac{1}{2}M\, \psi_{La}\epsilon_{ab}\psi_{Lb} + \tfrac{1}{2}M\, \psi^*_{La}\epsilon_{ab}\psi^*_{Lb} \right\}$$

$$i\frac{\partial}{\partial x^0}\psi_{La} = [\psi_{La}(x), H]$$

$$= \left[\psi_{La}(x), \int d^3y \left\{ \psi^\dagger_{Lc}(y)\, i\vec{\sigma}_{cd}\cdot\vec{\nabla}\, \psi_{Ld}(y) + \tfrac{1}{2}M\, \psi^*_{Lc}\epsilon_{cd}\psi^*_{Ld} \right\} \right]$$

$$= \int d^3y \Big[\{\psi_{La}(x), \psi^\dagger_{Lc}(y)\} (i\sigma\nabla\psi_L)_c$$
$$+ \tfrac{1}{2}M \{\psi_{La}(x)\, \psi^*_{Lc}\}\epsilon_{cd}\psi^*_{Ld} - \tfrac{1}{2}M\, \psi^*_{Lc}\epsilon_{cd}\{\psi_{La}(x), \psi^\dagger_{Ld}(y)\} \Big]$$

$$= (i\vec{\sigma}\cdot\vec{\nabla}\psi_L)_a + \tfrac{1}{2}M\epsilon_{ad}\psi^*_{Ld} - \tfrac{1}{2}M\,\psi^*_{Lc}\epsilon_{ca}$$

$$= (i\vec{\sigma}\cdot\vec{\nabla}\psi_L)_a + M\epsilon_{ab}\psi^*_{Lb}$$

then the Majorana equation is

$$i\bar{\sigma}^\mu_{ab}\partial_\mu\psi_{Lb} - M\epsilon_{ab}\psi^*_{Lb} = 0$$

To square this equation, first note that

$$\sigma^\mu\partial_\mu\, \bar{\sigma}^\mu\partial_\mu = \partial_0^2 - \vec{\nabla}^2 = \partial^\mu\partial_\mu$$

and

$$\sigma^\mu\partial_\mu = (i\sigma^2)[(\bar{\sigma}^\mu)^*\partial_\mu](-i\sigma^2)$$

$$\epsilon_{ab} = (i\sigma^2)_{ab}$$

$$(i\sigma\cdot\partial)(i\bar{\sigma}\cdot\partial)\psi_L$$

$$= (i\sigma^2)(i(\bar{\sigma})^*\cdot\partial)(-i\sigma^2)(i\bar{\sigma}\cdot\partial)\psi_L$$

$$= (i\sigma^2)(i(\bar{\sigma}\cdot)^*\partial)(-i\sigma^2)(i\sigma^2)M\psi_L^*$$

$$= (i\sigma^2)\; i\bar{\sigma}^*\cdot\partial\; M\,\psi_L^*$$

$$= (i\sigma^2)\, M\;\; i\bar{\sigma}^*\partial\psi_L^*$$

$$= (i\sigma^2)\, M\,(-M(i\sigma^2)\psi_L)$$

$$= \quad M^2\psi_L$$

then $\quad(-\partial^2 - M^2)\psi_L = 0$

so ψ_L is a quantum field for particles of mass M.
These particles are linear combination of the particles and antiparticles of the original field ψ_L.