Physics 330 – Problem Set # 2

Solutions

1.) The solutions $u(p)$ satisfy the equation

$$(\not{p} - m)\, u(p) = 0$$

for $p_0 = (m, \vec{0})$ this is

$$\left[\left(\begin{array}{c|c} & m \\ \hline m & \end{array} \right) - \left(\begin{array}{c|c} m & \\ \hline & m \end{array} \right) \right] \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0$$

$$m u_R - m u_L = 0$$
$$m u_L - m u_R = 0$$

so $u_L = u_R$ & there are 2 solutions

$$u_L = u_R \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_L = u_R \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

adopting the normalization used in class

$$u_+ = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_- = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

the angular momentum operator acting on u is

$$J^k = \frac{1}{2}\left(\begin{array}{c|c} \sigma^k & \\ \hline & \sigma^k \end{array}\right) \quad , \quad J^3 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right)$$

so
$$J^3 u_+(p) = +\frac{1}{2} u_+(p)$$
$$J^3 u_-(p) = -\frac{1}{2} u_-(p)$$

b) the boost operator on spinors is

$$\Lambda_{\frac{1}{2}}(\eta) = \exp\left[-\frac{\eta^k}{2}\left(\begin{array}{c|c} \sigma^k & \\ \hline & -\sigma^k \end{array}\right)\right]$$

For a boost along $\hat{3}$

$$\Lambda_{\frac{1}{2}} = \exp\left[\left(\begin{array}{c|c} -\eta/2\,\sigma^3 & 0 \\ \hline 0 & +\eta/2\,\sigma^3 \end{array}\right)\right]$$

$$= \cosh\eta/2 + \sinh\eta/2 \left(\begin{array}{c|c} -\sigma^3 & \\ \hline & \sigma^3 \end{array}\right)$$

where $\cosh\eta/2 = \left(\frac{E_p + p}{m}\right)^{\frac{1}{2}}$ $\quad \sinh\eta/2 = \left(\frac{E_p - p}{m}\right)^{\frac{1}{2}}$ $\quad \cosh^2\eta/2 - \sinh^2\eta/2 = 1$ ✓

Then

$$u_+(p) = \Lambda_{\frac{1}{2}}(\eta)\, u_+(p_0) = \sqrt{m}\left(\begin{array}{c} e^{-\eta/2}\binom{1}{0} \\ \hline e^{+\eta/2}\binom{1}{0} \end{array}\right) = \left(\begin{array}{c} (E-p)^{\frac{1}{2}}\binom{1}{0} \\ \hline (E+p)^{\frac{1}{2}}\binom{1}{0} \end{array}\right)$$

$$u_-(p) = \Lambda_{\frac{1}{2}}(\eta)\, u_-(p_0) = \sqrt{m}\left(\begin{array}{c} e^{+\eta/2}\binom{0}{1} \\ \hline e^{-\eta/2}\binom{0}{1} \end{array}\right) = \left(\begin{array}{c} (E+p)^{\frac{1}{2}}\binom{0}{1} \\ \hline (E-p)^{\frac{1}{2}}\binom{0}{1} \end{array}\right)$$

$$J^3 = \frac{1}{2}\left(\begin{array}{c|c} \sigma^3 & \\ \hline & \sigma^3 \end{array}\right) \quad \text{so} \quad J^3 u_+ = +\tfrac{1}{2} u(p)$$

$$J^3 u_- = -\tfrac{1}{2} u(p)$$

c.) The rotation operator needed is

$$\Lambda_{\frac{1}{2}} = \exp\left[-i\theta J^2\right] = \exp\left[-i\theta_{/2}\left(\begin{array}{c|c} \sigma^2 & \\ \hline & \sigma^2 \end{array}\right)\right]$$

$$= \exp\left[\begin{array}{cc|cc} & -\theta/2 & & \\ \theta/2 & & & \\ \hline & & & -\theta/2 \\ & & \theta/2 & \end{array}\right] = \left[\begin{array}{cc|cc} \cos\theta/2 & -\sin\theta/2 & & \\ \sin\theta/2 & \cos\theta/2 & & \\ \hline & & \cos\theta/2 & -\sin\theta/2 \\ & & \sin\theta/2 & \cos\theta/2 \end{array}\right]$$

$$\Lambda_{\frac{1}{2}} u_+(p) = \left[\begin{array}{c} (E_p-p)^{1/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix} \\ \hline (E_p+p)^{1/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}\end{array}\right] \qquad \Lambda_{\frac{1}{2}} u_-(p) = \left[\begin{array}{c} (E_p+p)^{1/2}\begin{pmatrix}-\sin\theta/2 \\ \cos\theta/2\end{pmatrix} \\ \hline (E_p-p)^{1/2}\begin{pmatrix}-\sin\theta/2 \\ \cos\theta/2\end{pmatrix}\end{array}\right]$$

d.) $$p = (E_p, p\sin\theta, 0, p\cos\theta)$$

$$\gamma^\mu p_\mu = \left(\begin{array}{c|c} & \sigma^\mu p_\mu \\ \hline \bar{\sigma}^\mu p_\mu & \end{array}\right) = \left(\begin{array}{c|c} & E_p - \vec{\sigma}\cdot\vec{p} \\ \hline E_p + \vec{\sigma}\cdot\vec{p} & \end{array}\right) = \left(\begin{array}{cc|cc} & & E_p - p^3 & -p^1 \\ & & -p^1 & E_p+p^3 \\ \hline E_p+p^3 & p^1 & & \\ p^1 & E_p-p^3 & & \end{array}\right)$$

acting on $u_+(p)$

$$\begin{pmatrix} E_p - p^3 & -p^1 \\ -p^1 & E_p+p^3 \end{pmatrix}(E_p+p)^{1/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix} = (E_p+p)^{1/2}\begin{bmatrix} E_p\cos\theta/2 - p\cos\theta\cos\theta/2 - p\sin\theta\sin\theta/2 \\ -p\sin\theta\cos\theta/2 + E_p\sin\theta/2 + p\cos\theta\sin\theta/2 \end{bmatrix}$$

$$= (E_p+p)^{1/2}\begin{bmatrix} E_p\cos\theta/2 - p\cos\theta/2 \\ -p\sin\theta/2 + E\sin\theta/2 \end{bmatrix} = (E_p+p)^{1/2}\begin{pmatrix} (E_p-p)\cos\theta/2 \\ (E_p-p)\sin\theta/2 \end{pmatrix}$$

$$= \left[E_p^2 - p^2\right]^{1/2}(E_p-p)^{1/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}$$

$$= m\,(E_p-p)^{1/2}\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}$$

$$\begin{pmatrix} E_p+p^3 & p^1 \\ p^1 & E_p-p^3 \end{pmatrix} (E_p-p)^{1/2} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$= (E_p-p)^{1/2} \begin{bmatrix} E_p\cos\theta/2 + p\cos\theta\cos\theta/2 + p\sin\theta\sin\theta/2 \\ p\sin\theta\cos\theta/2 + E_p\sin\theta/2 - p\cos\theta\sin\theta/2 \end{bmatrix}$$

$$= (E_p-p)^{1/2} \begin{bmatrix} E_p\cos\theta/2 + p\cos\theta/2 \\ E_p\sin\theta/2 + p\sin\theta/2 \end{bmatrix} = (E_p-p)^{1/2}(E_p+p)\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$= m(E_p+p)^{1/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

then

$$\left[\begin{pmatrix} & \sigma\cdot p \\ \bar{\sigma}\cdot p & \end{pmatrix} - \begin{pmatrix} m & \\ & m \end{pmatrix}\right]\begin{bmatrix} (E_p-p)^{1/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \\ (E_p+p)^{1/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \left[m(E_p-p)^{1/2} - m(E_p-p)^{1/2}\right]\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \\ m(E_p+p)^{1/2} - m(E_p+p)^{1/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \end{bmatrix} = 0$$

acts on $u_-(p)$

$$\begin{pmatrix} E_p-p^3 & -p^1 \\ -p^1 & E_p+p^3 \end{pmatrix}(E_p-p)^{1/2}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$= (E_p-p)^{1/2}\begin{bmatrix} -E_p\sin\theta/2 + p\cos\theta\sin\theta/2 - p\sin\theta\cos\theta/2 \\ +p_1\sin\theta\sin\theta/2 + E_p\cos\theta/2 + p\cos\theta\cos\theta/2 \end{bmatrix}$$

$$= (E_p-p)^{1/2}\begin{bmatrix} -E_p\sin\theta/2 - p\sin\theta/2 \\ E_p\cos\theta/2 + p\sin\theta/2 \end{bmatrix} = (E_p-p)^{1/2}(E_p+p)\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$= m(E_p+p)^{1/2}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$\begin{pmatrix} E_p+p^3 & p^1 \\ p^1 & E_p-p^3 \end{pmatrix}(E_p+p)^{1/2}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$= (E_p+p)^{1/2}\begin{bmatrix} -E_p\sin\theta/2 + p\cos\theta\sin\theta/2 - p\sin\theta\cos\theta/2 \\ -p\sin\theta\sin\theta/2 + E_p\cos\theta/2 - p\cos\theta\cos\theta/2 \end{bmatrix}$$

$$= (E_p+p)^{1/2}\begin{bmatrix} -E_p\sin\theta/2 + p\sin\theta/2 \\ E_p\cos\theta/2 - p\sin\theta/2 \end{bmatrix} = (E_p+p)^{1/2}(E_p-p)\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$= m(E_p-p)^{1/2}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

then

$$\left[\begin{pmatrix} & \sigma\cdot p \\ \bar\sigma\cdot p & \end{pmatrix} - \begin{pmatrix} m & \\ & m \end{pmatrix}\right]\begin{pmatrix} (E_p+p)^{1/2}\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix} \\ (E_p-p)^{1/2}\begin{pmatrix}-\sin\theta/2 \\ \cos\theta/2\end{pmatrix}\end{pmatrix} = \begin{bmatrix} [m(E_p+p)^{1/2} - m(E_p+p)^{1/2}]\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix} \\ m(E_p-p)^{1/2} - m(E_p-p)^{1/2}]\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}\end{bmatrix}$$

$$= 0$$

so

$$(\not{p} - m)\, u_+(p) = 0 = (\not{p} - m)\, u_-(p)$$

e) to rotate about the $\hat{3}$ axis, the rotation matrix is

$$\Lambda_{1/2} = \exp\left(-i\phi J^3\right) = \begin{pmatrix} e^{-i\phi/2} & & & \\ & e^{+i\phi/2} & & \\ & & e^{-i\phi/2} & \\ & & & e^{+i\phi/2}\end{pmatrix}$$

then u_+ u_- rotated through ϕ are:

$$u_+ = \begin{pmatrix} (E_p-p)^{1/2}\begin{pmatrix}\cos\theta/2\; e^{-i\phi/2} \\ \sin\theta/2\; e^{+i\phi/2}\end{pmatrix} \\ (E_p+p)^{1/2}\begin{pmatrix}\cos\theta/2\; e^{-i\phi/2} \\ \sin\theta/2\; e^{+i\phi/2}\end{pmatrix}\end{pmatrix} \qquad u_- = \begin{pmatrix} (E_p+p)^{1/2}\begin{pmatrix}-\sin\theta/2\; e^{-i\phi/2} \\ \cos\theta/2\; e^{+i\phi/2}\end{pmatrix} \\ (E-p)^{1/2}\begin{pmatrix}-\sin\theta/2\; e^{-i\phi/2} \\ \cos\theta/2\; e^{+i\phi/2}\end{pmatrix}\end{pmatrix}$$

2) $\bar{u}(p')\gamma^\mu u(p)$

$$= \frac{1}{2m}\,\bar{u}(p')\left[\not{p}'\gamma^\mu + \gamma^\mu \not{p}\right] u(p)$$

$$\gamma^\mu \not{p} = \gamma^\mu \gamma^\nu p_\nu = \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} p_\nu + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] p_\nu$$

$$= \eta^{\mu\nu} p_\nu - i/_2\,(i[\gamma^\mu, \gamma^\nu])\,p_\nu$$

$$= p^\mu - i\sigma^{\mu\nu} p^\nu$$

$$\not{p}'\gamma^\mu = \tfrac{1}{2}\{\gamma^\nu, \gamma^\mu\} p'_\nu + \tfrac{1}{2}[\gamma^\nu, \gamma^\mu]\, p'_\nu$$

$$= p'^\mu + i\,\tfrac{i}{2}[\gamma^\mu, \gamma^\nu]\, p'_\nu$$

so

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\,\bar{u}(p')\left[(p'+p)^\mu + i\,\sigma^{\mu\nu}(p'-p)_\nu\right] u(p)$$

<u>QED</u>

3.) $$\left[i\gamma^\mu(\partial_\mu+ieA_\mu)+m\right]\left[i\gamma^\nu(\partial_\nu+ieA_\nu)-m\right]$$

$$= -\gamma^\mu\gamma^\nu(\partial_\mu+ieA_\mu)(\partial_\nu+ieA_\nu) - m^2$$

$$= \Big(-\tfrac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{\eta^{\mu\nu}} - \tfrac{1}{2}\underbrace{[\gamma^\mu,\gamma^\nu]}_{-i\sigma^{\mu\nu}}\Big)(\partial_\mu+ieA_\mu)(\partial_\nu+ieA_\nu) - m^2$$

$$= -(\partial_\mu+ieA_\mu)^2 + i\sigma^{\mu\nu}(\partial_\mu+ieA_\mu)(\partial_\nu+ieA_\nu) - m^2$$

$$= \underbrace{-(\partial_\mu+ieA_\mu)^2-m^2}_{\substack{\text{Klein Gordon equation}\\ \text{coupled to } A_\mu}} + \underbrace{\tfrac{i}{2}\sigma^{\mu\nu}[\partial_\mu+ieA_\mu,\ \partial_\nu+ieA_\nu]}_{\text{extra term}}$$

$$= -(\partial_\mu+ieA_\mu)^2-m^2 + \tfrac{i}{2}\sigma^{\mu\nu}(ie)(\partial_\mu A_\nu-\partial_\nu A_\mu)$$

$$= -(\partial_\mu+ieA_\mu)^2-m^2 - e\frac{\sigma^{\mu\nu}}{2}F_{\mu\nu}$$

where $F_{\mu\nu}$ is the electromagnetic field strength, eg $F_{ij} = \epsilon_{ijk}B^k$

In a static B field

$$= -(\partial_\mu+ieA_\mu)^2-m^2 - e\frac{\varepsilon_{ijk}\sigma^{ij}}{2}B^k$$

$$\sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j] = \frac{i}{2}\left[\begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}\begin{pmatrix} & \sigma^j \\ -\sigma^j & \end{pmatrix} - \begin{pmatrix} & \sigma^j \\ -\sigma^j & \end{pmatrix}\begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}\right]$$

$$= \frac{-i}{2}\begin{bmatrix} \sigma^i\sigma^j-\sigma^j\sigma^i & \\ & \sigma^i\sigma^j-\sigma^j\sigma^i \end{bmatrix}$$

$$= \frac{-i}{2}\,2i\epsilon^{ijl}\begin{bmatrix}\sigma^l & \\ & \sigma^l\end{bmatrix} = \epsilon^{ijl}\begin{bmatrix}\sigma^l & \\ & \sigma^l\end{bmatrix}$$

then our operator is

$$\left[-(\partial_\mu + ieA_\mu)^2 - m^2\right] - e\,\frac{1}{2}\,\epsilon^{ij\ell}\begin{pmatrix}\sigma^\ell & \\ & \sigma^\ell\end{pmatrix}\epsilon_{ijk}\,B^k$$

$$= \left[-(\partial_\mu + ieA_\mu)^2 - m^2\right] - e\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}\cdot B^k$$

For a Dirac particle at rest, the energy depends on the <u>spin</u> via

$$\left[m^2 + e\begin{pmatrix}\vec{\sigma} & \\ & \vec{\sigma}\end{pmatrix}\cdot\vec{B} + \cdots\right]^{1/2}$$

$$= m + \frac{e}{2m}\begin{pmatrix}\vec{\sigma} & \\ & \vec{\sigma}\end{pmatrix}\cdot B + \cdots$$

$$= m + 2\cdot\frac{e}{2m}\,\vec{S}\cdot\vec{B} \qquad \vec{S} = \begin{pmatrix}\vec{\sigma}/2 & \\ & \vec{\sigma}/2\end{pmatrix}$$

so

$$\Delta E = -2\,\frac{(-e)}{2m}\,\vec{S}\cdot\vec{B}$$

This is the coupling to an external magnetic field for a particle with $q = (-e)$ and $g = 2$