Physics 330 – Problem Set # 1

Solution

1.) a.) For $\mathcal{H} = \int d^3z \left\{ \pi^\dagger \pi + \vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi + m^2 \Phi^\dagger \Phi \right\}$

$$i \frac{\partial}{\partial t} \Phi(x) = \left[\Phi(x), \int d^3z \; \pi^\dagger(z) \pi(z) \right]$$

$$= \int d^3z \; \left[\Phi(x), \pi^\dagger(z) \right] \pi(z) = i \int d^3z \; \delta^{(3)}(x-z) \, \pi(z)$$

$$= i \pi(x)$$

similarly,

$$i \frac{\partial}{\partial t} \pi(x) = \left[\pi(x), \int d^3z \left\{ \vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi + m^2 \Phi^\dagger \Phi \right\} \right]$$

$$= \left[\pi(x), \int d^3z \left\{ \Phi^\dagger(z)(-\nabla^2 \Phi) + m^2 \Phi^\dagger \Phi \right\} \right]$$

$$= \int d^3z \left\{ \left[\pi(x), \Phi^\dagger(z) \right] (-\nabla^2 \Phi(z)) + m^2 \left[\pi(x), \Phi^\dagger(z) \right] \Phi(z) \right\}$$

$$= (-i) \left\{ -\nabla^2 \Phi(x) + m^2 \Phi(x) \right\}$$

that is

$$\frac{\partial}{\partial t} \Phi = \pi \qquad \frac{\partial}{\partial t} \pi = + \nabla^2 \Phi - m^2 \Phi$$

or

$$\left(\frac{\partial^2}{\partial t^2} \Phi - \nabla^2 \Phi + m^2 \Phi \right) = 0$$

In the same way,

$$i\frac{\partial}{\partial t}\Phi^\dagger(x) = \left[\Phi^\dagger(x), \int d^3z\, \Pi^\dagger(z)\Pi(z)\right] = \int d^3z\, \Pi^\dagger(z)\left[\Phi^\dagger(x), \Pi(z)\right]$$

$$= i\Pi^\dagger(x)$$

$$i\frac{\partial}{\partial t}\Pi^\dagger(x) = \left[\Pi^\dagger(x), \int d^3z \left\{\vec{\nabla}\Phi^\dagger\cdot\vec{\nabla}\Phi + m^2\Phi^\dagger\Phi\right\}\right]$$

$$= \int d^3z \left\{-\nabla^2\Phi^\dagger\, [\Pi^\dagger(x), \Phi(z)] + m^2\Phi^\dagger(z)\, [\Pi^\dagger(x), \Phi(z)]\right\}$$

$$= (-i)\left(-\nabla^2\Phi^\dagger(x) + m^2\Phi^\dagger(x)\right)$$

so $$\frac{\partial^2}{\partial t^2}\Phi^\dagger(x) = +\nabla^2\Phi^\dagger(x) - m^2\Phi^\dagger(x)$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\Phi^\dagger(x) = 0$$

b.) $$\Phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left(a_p + b^\dagger_{-p}\right) \qquad \Phi^\dagger(x) = \int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left(b_p + a^\dagger_{-p}\right)$$

we can guess

$$\Pi(x) = \int\frac{d^3p}{(2\pi)^3}(-iE_p)\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left(a_p - b^\dagger_{-p}\right)$$

then $$\Pi^\dagger(x) = \int\frac{d^3p}{(2\pi)^3}(-iE_p)\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left(b_p - a^\dagger_{-p}\right)$$

since a's and b's commute and a_p's commute with a_p's, etc.

$$[\Phi(\vec{x}), \Pi(\vec{y})] = 0 = [\Phi^\dagger(x), \Pi^\dagger(y)]$$

now compute:

$$[\Phi(x), \Pi^{+}(y)]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} (-iE_q) \frac{1}{\sqrt{2E_p\, 2E_q}} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \left([a_p, -a^{+}_{-q}] + [b^{+}_{-p}, b_q]\right)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-i\,E_q}{\sqrt{2E_p\, 2E_q}} e^{i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}} \left(-(2\pi)^3 \delta^{(3)}(\vec{p}+\vec{q})\right) \cdot 2$$

$$= \int \frac{d^3p}{(2\pi)^3} (-i) \frac{E_p}{2E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (-2) = i\delta^{(3)}(\vec{x}-\vec{y})$$

the calculation of $[\Phi^{+}(x), \Pi(y)]$ is exactly the same with

$$a \Leftrightarrow b$$

c.) Plug the formulae from part b ~~into~~ $\mathcal{H}$

$$\int d^3z\, \Pi^{+}\Pi = \int d^3z \int \frac{d^3p\, d^3q}{(2\pi)^6} e^{i\vec{p}\cdot\vec{z}} e^{i\vec{q}\cdot\vec{z}} (b_p - a^{+}_{-p})(a_q - b^{+}_{-q}) \times \left(-\frac{E_pE_q}{\sqrt{2E_p\, 2E_q}}\right)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} (2\pi)^3 \delta(\vec{p}+\vec{q}) \times \left(-\frac{E_pE_q}{(2E_p\, 2E_q)^{1/2}}\right) \times \left(b_p a_q - b_p b^{+}_{-q} - a^{+}_{-p} a_q + a^{+}_{-p} b^{+}_{-q}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} \left[-b_p a_{-p} + (b^{+}_p b_p + (\text{const})) + a^{+}_{-p} a_{-p} - a^{+}_{-p} b^{+}_p \right]$$

similarly,

$$\int d^3z\, m^2 \Phi^{+}\Phi = \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{2E_p} \left(b_p a_{-p} + (b^{+}_p b_p + (\text{const})) + a^{+}_{-p} a_{-p} + a^{+}_{-p} b^{+}_{-p} \right)$$

$$\int d^3z \quad \vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi$$

$$= \int d^3z \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{i\vec{p} \cdot i\vec{q}}{\sqrt{2E_p\, 2E_q}} e^{i(p+q)\cdot z}$$

$$\cdot \left(b_p a_q + b_p b^\dagger_{-q} + a^\dagger_{-p} a_q + a^\dagger_{-p} b^\dagger_{-q} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{|\vec{p}|^2}{2E_p} \left(b_p a_{-p} + \left(b^\dagger_p b_p + (\text{const}) \right) + a^\dagger_{-p} a_{-p} + a^\dagger_{-p} b^\dagger_p \right)$$

then $\int d^3z \left(\vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi + m^2 \Phi^\dagger \Phi \right)$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{(m^2 + |\vec{p}|^2)}{2E_p} \left(b_p a_{-p} + b^\dagger_p b_p + (\text{const}) + a^\dagger_{-p} a_{-p} + a^\dagger_{-p} b^\dagger_p \right)$$

adding this to $\int d^3z\ \Pi^\dagger \Pi$, with $|\vec{p}|^2 + m^2 = E_p^2$, the ba and $a^\dagger b^\dagger$ terms cancel. Dropping also the (const) terms, we find

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} E_p \left(a^\dagger_p a_p + b^\dagger_p b_p \right)$$

d) Compute Q in terms of a's and b's.

$$Q = \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} (-i)\, e^{i\vec{p}\cdot x} e^{iq\cdot x}$$

$$\times \left\{ \left(-i \frac{E_p}{\sqrt{2E_p\, 2E_q}} \right) \left(b_p - a^\dagger_{-p} \right) \left(a_q + b^\dagger_{-q} \right) \right.$$

$$\left. - \; -i \left(\frac{E_q}{\sqrt{2E_p\, 2E_q}} \right) \left(b_p + a^\dagger_{-p} \right) \left(a_q - b^\dagger_{-q} \right) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} (-i)(-i)\Big[\frac{E_p}{2E_p}\left(b_p a_{-p} + b_p b_p^\dagger - a_{-p}^\dagger a_{-p} - a_{-p}^\dagger b_p^\dagger\right)$$

$$- \frac{E_p}{2E_p}\left(b_p a_{-p} - b_p b_p^\dagger + a_{-p}^\dagger a_{-p} - a_{-p}^\dagger b_p^\dagger\right)\Big]$$

so

$$Q = \int \frac{d^3p}{(2\pi)^3}\left[a_p^\dagger a_p - b_p^\dagger b_p\right] + (\text{const.})$$

since the operators $a_p^\dagger a_p$ and $b_p^\dagger b_p$ commute with one another, we see immediately that $[Q, H] = 0$

e.) $$[Q, a_p^\dagger] = \left[\int \frac{d^3q}{(2\pi)^3} a_q^\dagger a_q \,,\, a_p^\dagger\right]$$

$$= \int \frac{d^3q}{(2\pi)^3} a_q^\dagger \,[a_q, a_p^\dagger]$$

$$= \int \frac{d^3q}{(2\pi)^3} a_q^\dagger \,(2\pi)^3 \delta(q-p)$$

$$= a_p^\dagger$$

similarly

$$[Q, a_p] = \int \frac{d^3q}{(2\pi)^3} [a_q^\dagger, a_p]\, a_q$$

$$= \int \frac{d^3q}{(2\pi)^3} \left[-(2\pi)^3 \delta(q-p)\right] a_q$$

$$= -a_p$$

We can rewrite these equations as

$$Q\, a_p^+ = a_p^+ (Q+1) \qquad Q\, a_p = a_p (Q-1)$$

That is, if $Q|\psi\rangle = \eta|\psi\rangle$ then

$a_p^+|\psi\rangle$ has charge $(\eta+1)$

$a_p|\psi\rangle$ has charge $(\eta-1)$

The same relations apply to b_p^+ b_p except for the minus sign in Q

$$Q\, b_p^+ = b_p^+ (Q-1) \qquad Q\, b_p = b_p (Q+1)$$

2:) a.) In this problem, the Hamiltonian is the sum of n independent Klein-Gordon ~~Hamiltonians~~ Hamiltonians. For each one, our analysis of the Klein-Gordon equation applies. So for example

$$i\frac{\partial}{\partial t}\phi_j(x) = [\phi_j(x), H]$$

$$= \left[\phi_j(x),\ \int d^3z\ \tfrac{1}{2}\pi_j^2(z)\right] \quad \text{since all other terms in } H \text{ commute with } \phi_j$$

$$= \int d^3z\ \frac{1}{2}\left\{[\phi_j(x), \pi_j(z)]\ \pi_j(z) + \pi_j(z)\ [\phi_j(x), \pi_j(z)]\right\}$$

$$= \int d^3z\ \tfrac{1}{2}\delta(x-z)\, i\pi_j(z)\cdot 2 \ = i\pi_j(x)$$

and similarly

$$i\frac{\partial}{\partial t}\pi_j(x) = [\pi_j(x), H]$$

$$= \int d^3z \left\{\left[\pi_j(x), \tfrac{1}{2}(\nabla\phi_j)^2\right] + \left[\pi_j(x), \tfrac{1}{2}m^2\phi_j^2\right]\right\}$$

$$= (-i)\left[-\nabla^2\phi_j(x) + m^2\phi_j(x)\right]$$

so

$$\frac{\partial^2}{\partial t^2}\phi_j(x) = \frac{\partial}{\partial t}\pi_j = \nabla^2\phi_j - m^2\phi_j$$

so $\left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2\right)\phi_j = 0$ for each j

b.) Let $[a_{jp}, a^{\dagger}_{kq}] = \delta_{jk} \cdot (2\pi)^3 \delta^{(3)}(p-q)$

then we can represent

$$\phi_j(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{jp} + a^{\dagger}_{j-p}) e^{i\vec{p}\cdot\vec{x}}$$

$$\pi_j(x) = \int \frac{d^3p}{(2\pi)^3} \frac{-iE_p}{\sqrt{2E_p}} (a_{jp} - a^{\dagger}_{j-p}) e^{i\vec{p}\cdot\vec{x}}$$

Following exactly for each value of j the steps in our analysis of the Klein-Gordon theory with one field $\phi(x)$, we find

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} E_p \sum_j a^{\dagger}_{jp} a_{jp} + (\text{const})$$

c.) $$Q^{\dagger}_{jk} = \int d^3x \, (\phi^{\dagger}_j \pi^{\dagger}_k - \phi^{\dagger}_k \pi^{\dagger}_j)$$

and since $\phi^{\dagger}_j = \phi_j$ $\quad \pi^{\dagger}_k = \pi_k$ and $[\phi_k, \pi_j] = 0$ for $j \neq k$

$$= \int d^3x \, (-\pi_j \phi_k + \pi_k \phi_j)$$

$$[Q_{jk}, H] = \int d^3x \, d^3y \left[(-\pi_j \phi_k + \pi_k \phi_j), \sum_\ell \left(\tfrac{1}{2}\pi_\ell^2 + \tfrac{1}{2}(\vec{\nabla}\phi_\ell)^2 + \tfrac{1}{2}m^2\phi_\ell^2\right) \right]$$

$$= \int d^3x \, d^3y \sum_\ell \Big([-\pi_j \phi_k, \tfrac{1}{2}\pi_\ell^2] + [-\pi_j \phi_k, \tfrac{1}{2}(\nabla\phi_\ell)^2 + \tfrac{1}{2}m^2\phi_\ell^2] - j \leftrightarrow k \Big)$$

$$= \int d^3x \, d^3y \sum_\ell \Big(-\pi_j \, i\delta_{k\ell} \delta^{(3)}(x-y) \pi_\ell - (-i\delta_{j\ell}(-\nabla^2\phi_\ell + m^2\phi_\ell)\delta(x-y))\phi_k - (j-k) \Big)$$

$$= \int d^3x \left[\left(-i\, \pi_j \pi_k + i\left(-\nabla^2\phi_j + m^2\phi_j\right)\phi_k\right) - (j-k)\right]$$

$$= \int d^3x \left[\left\{ -i\, \pi_j \pi_k + i\, \vec\nabla\phi_j \vec\nabla\phi_k + i m^2 \phi_j \phi_k \right\} - (j \leftrightarrow k) \right]$$

but, the first line here is symmetric under $(j \leftrightarrow k)$!

$$= 0$$

d) Using the representations on p. 8

$$Q_{jk} = \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6}\, e^{i(\vec p + \vec q)\cdot\vec x}\, (-iE_p) \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} \left(-a_{jp} + a^\dagger_{j-p}\right)\left(a_{kq} + a^\dagger_{k-q}\right) - (j \leftrightarrow k)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{(-iE_p)}{2E_p} \left[-a_{jp} a_{k-p} - a_{jp} a^\dagger_{kp} + a^\dagger_{j-p} a_{k-p} + a^\dagger_{j-p} a^\dagger_{kp} \right] - (j \leftrightarrow k) \right)$$

Now

$$\int d^3p\, a_{jp} a_{k-p} \underset{\vec p \to -\vec p}{=} \int d^3p\, a_{j-p} a_{kp} \underset{\substack{\text{operators commute}\\ \text{for } k \neq j}}{=} \int d^3p\, a_{kp} a_{j-p}$$

so the first term in brackets is symmetric in $j \leftrightarrow k$

by the same logic, the $a^\dagger_{j-p} a^\dagger_{kp}$ term is symmetric in $j \leftrightarrow k$

also $[a_{jp}, a^\dagger_{kp}] \propto \delta_{jk}$ is symmetric so we can drop all 3 of these terms. What is left is

$$Q_{jk} = \int \frac{d^3p}{(2\pi)^3} \left(\frac{-i}{2}\right)\left(a^\dagger_{jp} a_{kp} - a^\dagger_{kp} a_{jp}\right) - (j \leftrightarrow k)$$

or

$$Q_{jk} = (-i) \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger_{jp} a_{kp} - a^\dagger_{kp} a_{jp} \right)$$

e.)

$$[Q_{jk}, Q_{lm}] = (-i)^2 \int \frac{d^3p\, d^3q}{(2\pi)^6} \Big(\big([a^\dagger_{jp} a_{kp}, a^\dagger_{lq} a_{mq}] - (j \leftrightarrow k) \big) - (l \leftrightarrow m) \Big)$$

Let's first work out

$$[a^\dagger_{jp} a_{kp}, a^\dagger_{lq} a_{mq}] = a^\dagger_{jp} [a_{kp}, a^\dagger_{lq}] a_{mq} + a^\dagger_{lq} [a^\dagger_{jp}, a_{mq}] a_{kp}$$

$$= (2\pi)^3 \delta(\vec{p} - \vec{q}) \left[\delta_{kl} (a^\dagger_{jp} a_{mq}) - \delta_{jm} (a^\dagger_{lq} a_{kp}) \right]$$

Now antisymmetrize:

$$[Q_{jk}, Q_{lm}] = -i \cdot (-i) \cdot \int \frac{d^3p}{(2\pi)^3}$$

$$\begin{aligned}
\Big[\; & \delta_{kl}\, a^\dagger_{jp} a_{mp} - \delta_{jm}\, a^\dagger_{lp} a_{kp} \\
& - \delta_{jl}\, a_{kp} a_{mp} + \delta_{km}\, a^\dagger_{lp} a_{jp} \\
& - \delta_{km}\, a^\dagger_{jp} a_{lp} + \delta_{jl}\, a^\dagger_{mp} a_{kp} \\
& + \delta_{jm}\, a^\dagger_{kp} a_{lp} - \delta_{kl}\, a^\dagger_{mp} a_{jp} \Big]
\end{aligned}$$

Hey! the right-hand side organizes itself into Q's.

$$[Q_{jk}, Q_{lm}] = (-i)\cdot(-i)\int\frac{d^3p}{(2\pi)^3}$$

$$\begin{aligned}[&\delta_{kl}(a^+_{jp}a_{mp} - a^+_{mp}a_{jp}) \\ &- \delta_{jl}(a^+_{kp}a_{mp} - a^+_{mp}a_{kp}) \\ &- \delta_{km}(a^+_{jp}a_{lp} - a^+_{lp}a_{jp}) \\ &+ \delta_{jm}(a^+_{kp}a_{lp} - a^+_{lp}a_{kp})]\end{aligned}$$

$$\begin{aligned}[Q_{jk}, Q_{lm}] = (-i)[&\delta_{kl}Q_{jm} - \delta_{jl}Q_{km} \\ &- \delta_{km}Q_{jl} + \delta_{jm}Q_{kl}]\end{aligned}$$

f) For $n=3$ there are 3 nonzero commutators:

$$[Q_{12}, Q_{23}] = -i\,Q_{13}$$

$$[Q_{23}, Q_{13}] = +i\,Q_{21}$$

$$[Q_{13}, Q_{12}] = +i\,Q_{32}$$

write $\mathcal{Q}_1 = Q_{23}$ $\quad\mathcal{Q}_2 = Q_{31}$ $\quad\mathcal{Q}_3 = Q_{12}$

these become

$$[\mathcal{Q}_3, \mathcal{Q}_1] = i \cdot \mathcal{Q}_2$$
$$[\mathcal{Q}_1, -\mathcal{Q}_2] = -i\mathcal{Q}_3$$
$$[-\mathcal{Q}_2, \mathcal{Q}_3] = -i\mathcal{Q}_1$$

so

$$[\mathcal{Q}_1, \mathcal{Q}_2] = i\mathcal{Q}_3 \qquad [\mathcal{Q}_2, \mathcal{Q}_3] = i\mathcal{Q}_1 \qquad [\mathcal{Q}_3, \mathcal{Q}_1] = i\mathcal{Q}_2$$

or $[\mathcal{Q}_a, \mathcal{Q}_b] = i\epsilon_{abc}\mathcal{Q}_c$

this is the commutation relations of angular momentum!

g.) The angular momentum operators are the generators of rotations in 3-dimensional space. In our current notation

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{\mathcal{Q}}}$$

Since any three of the Q_{jk} have this algebra, they generate rotations in a 3-dimensional subspace of the n-dimensional space of ϕ_j. Then the Q_{jk} must be the generators of n-dimensional rotations in this space

Q_{jk} generates rotations in the (j-k) plane in this n-dimensional space